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D-branes in orientifolds and orbifolds and Kasparov KK-theory

H. García-Compeán,[ab] W. Herrera-Suárez,[b] B.A. Itzá-Ortiz[c] and O. Loaiza-Brito[a]

[a] *Centro de Investigación y de Estudios Avanzados del IPN, Unidad Monterrey PIIT, Autopista Monterrey-Aeropuerto Km 10, 66600 Apodaca, Nuevo León, México*

[b] *Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, P.O. Box 14-740, 07000 México D.F., México*

[c] *Centro de Investigación en Matemáticas, Universidad Autónoma del Estado de Hidalgo, Carretera Pachuca-Tulancingo Km. 4.5, Pachuca, Hidalgo 42184, México*

E-mail: compean@fis.cinvestav.mx, wherrs@fis.cinvestav.mx, itza@uaeh.edu.mx, oloaiza@fis.cinvestav.mx

Abstract: A classification of D-branes in Type IIB Op^- orientifolds and orbifolds in terms of Real and equivariant KK-groups is given. We classify D-branes intersecting orientifold planes from which are recovered some special limits such as the spectrum for D-branes on top of Type I Op^- orientifold and the bivariant classification of Type I D-branes. The gauge group and transformation properties of the low energy effective field theory living in the corresponding unstable D-brane system are computed by extensive use of Clifford algebras. Some speculations about the existence of other versions of KK-groups, based on physical insights, are proposed. In the orbifold case, some known results concerning D-branes intersecting orbifolds are reproduced and generalized. Finally, the gauge theory of unstable systems in these orbifolds is recovered.

Keywords: Tachyon Condensation, D-branes, Non-Commutative Geometry, M(atrix) Theories.

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Contents

1. Introduction

Topological methods in physics have always been relevant in order to describe static stable configuration of finite energy in field and string theories. D-branes have RR charge and they are source of RR fields. Both of them are classified by K-theory in all different theories. The recipe is that the K-theory is described by the classes of pairs of gauge bundles over the worldvolume of the D9-$\overline{\text{D9}}$ pair of the Type IIB string theory or on the non-BPS D9 of the Type IIA theory. This description classifies all lower dimensional D-branes coming from tachyon condensation called descendent branes. However the inverse process of constructing higher dimensional D-branes from the lowest dimensional unstable systems of D-instantons is also possible. This description is given by using K-homology.

The incorporation of the K-homology description in the context of Matrix theory was done in [1]. This was called the K-matrix theory and is based in the process involving configurations of non-BPS instanton in Type IIA string theory and D-instantons - anti D-instantons in Type IIB theory. From these configurations, higher-dimensional D-branes can be constructed and they are classified (through their worldvolumes) by the K-homology groups. The D-branes are described and thus represented by equivalence classes of Connes spectral triples (analytical data) used in noncommutative geometry. This equivalence is physically defined in terms of the gauge equivalence and the charge conservation. For different approaches of K-homology to D-brane classification see [2 − 5].

K-theory and K-homology are dual one of each other and it is compelling to use the Kasparov (complex) KK-theory, which is a generalization of both theories. This was done in [1] where the procedure of the construction of ascendent-descendent brane configuration was implemented on the product space-time $X \times Y$ with the world-volume of the unstable D-brane wrapped on Y. The D-branes are classified in a natural way by the groups $\mathrm{KK}^i(X, Y)$. In [6] it was shown that D-branes of the Type I theory are classified by (the real/Orthogonal) $\mathrm{KKO}^i(X, Y)$.

Similarly to K-homology, there are several approaches for the KK-theory application in describing D-brane physics [7, 8, 4]; but we will concentrate in the approach from [1].

Moreover, in the present paper we extend these results by showing that orientifolds are classified by the Real KK-group $\mathrm{KKR}^i(X, Y)$ and orbifolds by the equivariant KK-group $\mathrm{KK}^i_G(X, Y)$. In addition, we propose based on physical arguments, the existence of different versions of the KK bifunctor which; as far as the authors knowledge have not been discussed in the literature before. For all these theories, the spectrum is correctly obtained. We also give an application to exotic orientifolds.

This paper is organized as follows. In section 2 a brief account of the classification of D-branes through K-theory, K-homology and KK-theory is given. Section 3 is devoted to describe Dd-branes in orientifold backgrounds by using the Real KKR-theory. In here we find general formulas which involve the two cases $q \leq p$ and $p \leq q$. Here p is the dimension of the orientifold plane Op and q is spatial dimension of the unstable Dq-brane. Section 4 analyzes the theory on the unstable D-brane using the information provided by the Clifford algebras involved in the definition of the KKR bifunctor. To be more specific we will describe in some detail three important examples. The rest of the cases is summarized in a table. Section 5 is devoted to make a proposal for extending the classification of D-branes in orientifolds to other theories such as the Type IIB with Op^+ (quaternionic) and the IIB with O9^+ (with gauge group USp(32)) orientifold in the context of Kasparov KK-theory. At the end of this section, we explain an application of our formalism to exotic orientifolds. D-branes in orbifold singularities with KK$_G$-theory are discussed in section 6. Finally in section 7 we give our final remarks. Four appendices collect a series of formal results about KK-theory.

2. Classification of D-branes in orientifold planes

2.1 D-branes and K-theory

In Type II superstring theories D-branes are constructed as solitons on unstable systems either formed by pairs of brane-antibranes or by single unstable D-branes [9]. This means that any configuration of D-brane charges is realized as a gauge field configuration on a stack of (sufficiently) many D9-$\overline{\text{D9}}$ branes in Type IIB, or non-BPS D9-branes in Type IIA by open string tachyon condensation. This was interpreted as a way to classify Dd-brane charges by gauge bundles on the worldvolume of the D9-branes [10]. Hence, D-brane charges turn out to be elements of a group constructed from equivalence classes of vector bundles, namely K-theory.

In Type IIB theory Dd-brane charges are classified by the so called complex K-theory group, which is valuated on the transversal space (with respect to the unstable system[1]) to Dd. In particular, the K-theory group classifying a Dd-brane in Type IIB is given by KU($\mathbb{R}^{9-d}$) which renders the Dd-brane as a soliton constructed by the pair D9-$\overline{\text{D9}}$. One can instead consider a Dd-brane as a soliton constructed from an unstable system formed by Dq-branes ($q > d$). The groups classifying the corresponding vector bundles transversal to the Dd-brane worldvolume, in a nine-dimensional or q-dimensional unstable system, are isomorphic as expected from Bott periodicity and are given by KU($\mathbb{R}^{9-d}$) and $KU^{q-1}(\mathbb{R}^{q-d})$ respectively.

D-brane classification by K-theory is a little more elaborated once we introduce discrete actions on the background such as orientifolds or orbifolds. For instance, Ramond-Ramond (RR) fields on which D-branes in Type I theory are charged, have a smaller number of degrees of freedom due to the orientifold projection. This reduces the gauge group on the D-brane to be orthogonal or symplectic implying that D-branes are classified by orthogonal K-theory groups of the corresponding transversal spaces. Specifically, Dd-branes in Type I are classified by $KO(\mathbb{R}^{9-d})$ which points out the presence of non-BPS states carrying discrete

[1]Throughout this paper, what we refer as "K-theory group" is really the reduced K-theory group of the compactified space.

topological charge [11]. These Dd-branes can also be thought of as solitonic constructions from unstable pairs of D9-$\overline{\text{D9}}$ branes on top of an orientifold nine-plane $O9^-$. In a similar context as before, we can try to understand the construction of Type I D-branes from lower-dimensional unstable branes (which is justified since in general, super Yang-Mills theories in 9+1 dimensions are non-renormalizable). In fact, it is possible to condense open string tachyons from a pair of Dq-$\overline{\text{D}q}$ on top of the orientifold nine-plane in order to construct Dd-branes, which are classified by $KO^{q-1}(\mathbb{R}^{q-d})$ [12].

The situation becomes much more interesting by considering the presence of lower dimensional orientifolds Op^-. Classification of Dd-branes in such backgrounds was given in [13] and it strongly depends on which type of orientifold background we are taking into account. It turns out that for an orientifold with a positive squared involution ($\tau^2 = 1$) and $(-1)^{F_L} = 1$ (i.e., for $p = 1 \ mod \ 4$) the real K-theory group which classifies Dd-branes is $KR(\mathbb{R}^{9-p,p-d})$, where

$$\mathbb{R}^{9-p,p-d} = \left(\mathbb{R}^{9-p}/\Omega \cdot \mathcal{I}_{9-p}\right) \times \mathbb{R}^{p-d}, \tag{2.1}$$

is the transversal space to the Dd-brane. The world-sheet operator Ω inverts the orientation of the string while the involution $\mathcal{I}_{9-p}$ maps transversal coordinates to the orientifold x_i to $-x_i$. Notice that Dd-branes on top of an orientifold plane Op^- are obtained as well by pairs of D9-$\overline{\text{D9}}$ in which the open string tachyons have been condensed. The corresponding construction from lower (than nine) dimensional unstable systems will be studied in the next section.

So far we have reviewed constructions of Dd-branes from unstable Dq-branes ($q > d$). This means that each element of K-theory describes a lower-dimensional (than q) Dd-brane obtained by tachyon condensation from unstable branes generalizing the D-brane descent relations (see [9] and references therein).

However, it is also possible to elucidate the above construction from the tree-level action of an unstable brane. Such an action is constructed in the Boundary String Field Theory (BSFT) to the superstring [14]. For instance, the action of an unstable D9-brane in Type IIA is given by

$$S = T_9 \int d^{10}x \left((\ln\ 2)\alpha' e^{-T^2/4} \partial^\mu T \partial_\mu T + e^{-T^2/4} \right) \tag{2.2}$$

from which a solution for the equations of motion for the tachyon field is

$$T = \mu X, \tag{2.3}$$

where μ is a constant and X denotes some coordinate of the spacetime manifold.

By substituing this kink solution into the unstable D9-brane action, we get the action of a stable D8-brane (for $\mu \to \infty$). The argument can be generalized to show that from the action for N non-BPS D9-branes, with N large enough

$$T(X) = \mu \sum_{i=1}^{9-d} X^i \gamma_i, \tag{2.4}$$

where $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, is also a solution for the equations of motion, giving rise to a Dd-brane. Notice that this expression for the tachyon field corresponds to the Atiyah-Bott-Shapiro (ABS) construction (see for instance [10, 15]), which plays a relevant role in the classification of D-branes by K-theory.

For the case of an unstable pair of brane anti-brane, the complete tachyon field is given by

$$\mathbf{F} = \begin{pmatrix} 0 & T^\dagger \\ T & 0 \end{pmatrix} = \mu \sum_{i=1}^{9-d} X^i \begin{pmatrix} 0 & \gamma_i^\dagger \\ \gamma_i & 0 \end{pmatrix}. \tag{2.5}$$

Hence, roughly speaking, Dd-branes are constructed by tachyon condensation from higher-dimensional unstable branes and they are classified by the gauge bundles on their corresponding transversal spaces.

2.2 D-branes and K-homology

In the context of Matrix theory it is possible to construct Dd-branes not from higher-dimensional unstable brane systems, but from infinitely many lower-dimensional D-branes. The idea was developed in [16] in order to construct commutative D-branes, which turn out to be classified by K-homology [1] in the case where the lower dimensional D-branes are D-instantons. The basic idea is as follows: by taking T-duality (in the euclidean space) on the nine spatial coordinates, the action (2.2) for the non-BPS D9-brane in Type IIA, generalized to N D9-branes for N large enough, gives

$$S = T_{-1}\mathrm{Tr}_{\,N\times N}\left(e^{-T^2/4}(1 - c_1[\phi_\mu, T]^2 - c_2\pi^2[\phi_\mu, \phi_\nu]^2)\right), \tag{2.6}$$

which is the action for N D(-1)-branes and where ϕ_μ are scalar fields representing the transverse position as a function on the coordinates x^ν.

The corresponding equations of motion for the tachyon field have as a solution (provided $\mu^2 = 1/c_1$)

$$\begin{aligned} T &= \frac{2\pi\mu}{\alpha'^{1/2}}p, \\ \phi_0 &= \frac{1}{2\pi\alpha'^{1/2}}x, \phi_i = 0, \qquad (i = 1, \ldots, 9), \end{aligned} \tag{2.7}$$

where the operators x and p are identified with the transversal coordinates and momentum of the non-BPS D(-1)-branes. Plugging this tachyon kink solution (in momentum) back into the D(-1)-branes action provides a D0-brane action, whose position is specified by the fields $\phi_i = 0$. The argument can be generalized to construct higher-dimensional Dd-branes in Type IIB theory from an infinite number of D(-1)-$\overline{\mathrm{D}(-1)}$ pairs, in which case the tachyon and scalar fields are

$$\begin{aligned} T &= \mu \sum_{j=0}^{d} p_j \otimes \gamma^j, \\ \phi_i^{(1)} &= \phi_i^{(2)} = \frac{1}{2\pi\alpha'^{1/2}}x^i, \qquad (i = 0, \ldots, d), \end{aligned} \tag{2.8}$$

with $\mu \to \infty$, and γ^j being the $2^{[\frac{d}{2}]} \times 2^{[\frac{d}{2}]}$ gamma matrices in d dimensions. The superindices in ϕ stand for the instanton brane and antibrane, respectively [16].

It follows then, that the tachyon matrix F can also be written as

$$F = \mu \sum_{j=0}^{d} p_j \otimes \Gamma^j, \tag{2.9}$$

with

$$\Gamma^j = \begin{pmatrix} 0 & \gamma^{j\dagger} \\ \gamma^j & 0 \end{pmatrix}. \tag{2.10}$$

However, since the tachyon field T (which comes from the oriented string between the instanton brane-antibrane system), is not projected out by GSO projection, represented by the operator $(-1)^{F_L} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, the tachyon matrix F satisfies the self-dual condition $F = F^\dagger$. This fact plays an important role in the next the sections.

Let us however, return to the question of classification of Dd-branes created by brane-anti-brane instantons. As in the *usual* case of tachyon condensation from higher dimensional non-BPS D-branes, the construction of D-branes from unstable D(-1)-branes leads to their classification in terms of the so called K-homology $K_n(X)$ [1], which roughly speaking, is the dual to the K-theory group $K^n(X)$ in the sense that it has a natural pairing with the K-theory group. Instead of classifiying vector bundles on the transverse space to a Dd-brane as in K-theory, K-homology classifies vector bundles on the worldvolume of the extended Dd-branes constructed from unstable $D(-1)$-branes.[2] This is generalized to construct a Dd-brane from an unstable Dq-brane ($q < d$) with a tachyon configuration given by

$$F = \mu \sum_{j=q+1}^{d+q} p_j \otimes \Gamma^j. \tag{2.11}$$

2.3 Kasparov KK-theory

By virtue of the material revisited so far, it is then natural to combine the above two setups in order to construct a Dd-brane by a kind of combination of tachyon condensation from

[2]More precisely, the topological K-homology of any locally compact space X classifies triples (M, E, ϕ), where

- M is a compact $spin^c$-manifold without boundary.
- E is a complex vector bundle over M.
- $\phi\colon M \to X$ is an embedding of M in X.

The equivalence relations on the triples (M, E, ϕ) that define the K-homology of X have a nice physical interpretation in terms of D-brane processes. In fact the components of the triples (M, E, ϕ) are easily interpreted as the worldvolume manifold M of the D-brane, E is the Chan-Paton bundle on the worldvolume M of the D-brane and ϕ is the embedding of the D-brane worldvolume in the ambient spacetime X. For more details see [1, 3].

higher- and lower-dimensional D-branes. For branes in Type II theories, the extension was given in [1], together with a proposal to classify them.

In this scenario, a Dd-brane located in coordinates $x^0, \ldots, x^{q-s}, x^{q+1}, \ldots, x^{d+s}$ is constructed roughly speaking by tachyon condensation from an unstable Dq-brane located in coordinates $x^0, \ldots, x^q$ with a tachyon configuration given by

$$F = \mu \sum_{i=0}^{s} X^i \otimes \Gamma_i + \mu \sum_{j=q+1}^{d+q} p_j \otimes \Gamma^j. \tag{2.12}$$

The "part" of the Dd-brane localized inside the unstable Dq-brane is constructed by tachyon condensation as in Sen's descent relations, while the rest can be seen as constructed from unstable Dq-branes as in section 2.2.

It turns out that the relevant group which classifies Dd-branes constructed as in the above configuration is Kasparov KK-theory [17, 1]. Let us first of all briefly summarize some important aspects about KK-theory (see Apendix A for a more formal and detailed description).

KK-theory is a generalization of both K-theory and K-homology, in the sense that while both K-theory and K-homology are functors from the category of locally compact Hausdorff topological spaces to the category of abelian groups, (i.e. classify classes of vector bundles on the transverse space and in the worldvolume of a D-brane, respectively), KK-theory is a bifunctor between these categories[3] [17 − 20]. The bifunctor assigns to each pair (X, Y) of locally compact topological spaces some abelian group denoted $KK^{-n}(X, Y)$ for any integer n. Here X denotes the part of the worldvolume of the D-brane (extended outside the Dq-brane system) created from lower dimensional branes and Y is the worldvolume of the unstable Dq-brane from which a D-brane is created by tachyon condensation (as in the descent relations). Given such identification of the topological spaces X and Y, it is then expected to get some relations between KK-theory and both K-theory and K-homology groups. Indeed, if $X = \{pt\}$, it means we do not have a D-brane (extended in the transverse space of the Dq-brane system) created from lower dimensional branes. This implies that Dd-branes are entirely classified by K-theory. Then

$$KK^{-n}(pt, Y) = K^{-n}(Y). \tag{2.13}$$

Similarly, for a brane fully extended outside the unstable Dq-brane from which it was constructed (via condensation of a tachyon field as in eq. (2.12)), the space Y is the point-space implying that

$$KK^{-n}(X, pt) = K_n(X). \tag{2.14}$$

Now, as in the case of K-theory which is the set of equivalence classes of vector bundles, KK-theory is the set of equivalence classes of Kasparov triplets $(\mathcal{H}, \phi, T)$. In pedestrian

[3]In fact, all the K-functors mentioned above have as domain the full category of C^*-algebras which includes the category of locally compactHausdorff spaces as a subcategory by assigning to each locally compact Hausdorff space X the C^*-algebra of continuous $\mathbb{C}$-valued functions on X vanishing at infinity. Moreover, it can be shown that each commutative C^*-algebra is of this form, X being the space of characters of the algebra.

JHEP12(2008)007

words, $\mathcal{H}$ is the set of all Chan-Paton gauge fields living on the worldvolume of the unstable Dq-brane (ϕ and T are as usual the transversal position and tachyon fields). In this sense, a zero class representing the vacuum is gathered by a tachyon field T which condensates trivially (i.e., without a kink solution in momentum or spatial configurations) implying that $T^2 = 1$ (T has been normalized) and T and ϕ depending on non-conjugate position and momentum, i.e. $[T, \phi] = 0$. For the case in which the tachyon condensates in a non-trivial way, it is said that the triplet is non-trivial, representing a D-brane configuration in which the tachyon field configuration is given by eq. (2.12). Hence, KK-theory is the set of triplets which are equivalent up to the addition of a zero-class triplet. It is, as in the case of K-theory, an equivalence which preserves the RR charge. A formal presentation of KK-theory groups is given in appendix A. However, for a more detailed explanation about the interpretation of the elements defining the Kasparov modules and the equivalence relations involved in the definition of the KK-groups, the reader is referred to [1], in which a detailed discussion on some subtleties in the choice of the spacetime and the tachyon in the Kasparov modules is considered.

2.3.1 D-branes and KK-theory

Let us consider the simple case of a Dd-brane in Type IIB(A) string theory, constructed from unstable Dq-branes. In particular, for a configuration of a Dd-brane located in coordinates $x^0, \ldots, x^{q-s}$, $x^{q+1}, \ldots, x^{d+s}$, the spaces X and Y' are given by $\mathbb{R}^{d-q+s}$ and $\mathbb{R}^{9-q+s}$ from which the relevant KK-theory group is given by[4]

$$KK^{0(-1)}(\mathbb{R}^{d-q+s}, \mathbb{R}^{9-q+s}) = K^{0(-1)}(\mathbb{R}^{9-d}). \tag{2.15}$$

It is important to stress out that, as mentioned in appendix B, it is possible to extract information of the system through the relation with complexified Clifford algebras $\mathbb{C}l^n$ given by

$$KK^{-n}(X, Y) = KK(C_0(X), C_0(Y) \otimes \mathbb{C}l^n), \tag{2.16}$$

where $C_0(X)$ ($C_0(Y)$) denotes the algebra of complex valued (real valued when dealing with orthogonal KK-groups) continuous functions in X (Y) vanishing at infinity. Such relation with Clifford algebras shall become very important in our description of Dd-branes in more general backgrounds.

The next natural step is to classify Dd-branes in Type I theory, i.e., in the presence of a negative RR charged orientifold nine-plane $O9^-$. This was done in [6], where the authors proposed that the relevant group for such classification is the real Kasparov bifunctor,

[4]In (2.13) we interpreted Y as the worldvolume of the unstable Dq-brane for some integer n; but similar to K-theory, Y is also interpreted as the transverse space (with respect to the Dq-brane) of the part of the Dd-brane localized inside the Dq-brane. This is achieved by making use of the Atiyah-Bott-Shapiro construction in K-theory. Moreover a similar meaning is assigned to Y', i.e. is the transverse space of the part of the Dd-brane extended inside the unstable Dq-brane system, but in this case the transverse space is relative to an unstable D9 system and consequently n in (2.13) is changed depending on the string theory we are dealing with. The KK-theory prescription in terms of Y and Y' are equivalent as will be shown through out this paper.

JHEP12(2008)007

denoted as $KKO(X,Y)$, in which roughly speaking, all complex fields become real by the orientifold nine projection (for a formal description and for more details, see appendix B).

Let us consider the Dd-brane in an $O9^-$-plane background extended again in the coordinates $x^0,\ldots,x^{q-s}$, $x^{q+1},\ldots,x^{d+s}$. In this situation the Kasparov KK-theory group turns out to be orthogonal (real) given by $KKO(\mathbb{R}^{d-q+s},\mathbb{R}^{9-q+s})$. Using the isomorphisms from eq. (B.5), the above group reduces to

$$KKO^{q-1}(\mathbb{R}^{d+s-q},\mathbb{R}^{s}) = KO^{q-1}(\mathbb{R}^{q-d}) = KO(\mathbb{R}^{9-d}), \tag{2.17}$$

as expected [12]. The relation with real Clifford algebras $Cl^{*,*}$ is given in a similar context as in Type II

$$KKO^{q-1}(X,Y) = KKO(C_0(X), C_0(Y)\otimes Cl^{1,q}), \tag{2.18}$$

for which the tachyon configuration reads

$$F = u\sum_{\alpha=q-s+1}^{q} x_\alpha \otimes \Gamma^\alpha + u\sum_{\beta=q+1}^{d+s}(-i\partial_\beta)\otimes\Gamma^\beta, \tag{2.19}$$

where Γ^α and $-i\Gamma^\beta$ are in $M_n(\mathbb{R})\otimes Cl^{1,q}_{\text{odd}}$ for some n (see apendix B.2), satisfying

$$\begin{aligned} &\Gamma^{\alpha\dagger} = \Gamma^\alpha, && (-i\Gamma^\beta)^\dagger = i\Gamma^\beta, && \\ &\{\Gamma^\alpha,\Gamma^{\alpha'}\} = 2\delta^{\alpha,\alpha'}, && \{\Gamma^\beta,\Gamma^{\beta'}\} = 2\delta^{\beta,\beta'}, && \{\Gamma^\alpha,\Gamma^\beta\} = 0. \end{aligned} \tag{2.20}$$

One can note that many physical properties of D-branes are obtained through the analysis of Clifford algebras. Indeed, as it has been carefully studied in [6] for $q=9$ and $s=9-d$ and for $q=-1$, it is possible to extract some information as the tension of the Type I D-branes from the Type IIB ones and the gauge field representations of the tachyon associated to the worldvolume field theory of the Type I Dd-branes constructed from instantons. This is achieved by looking at the representation theory of the real Clifford algebras involved in the definition of the KKO-groups.

3. Dd-branes in orientifold backgrounds and real KKR-theory

Up to now, we have reviewed the classification of D-branes in terms of K-theory, K-homology and KK-theory. For instance, we have seen that the Real K-theory group KR is the correct one to classify D-branes constructed from non-BPS D9-branes in Type II orientifolds $O1, O5$ and $O9$. On the other hand, we have a classification of D-branes, constructed from non-BPS Dq-branes in Type I theory. The next thing to do is to classify Dd-branes by KK-theory in a more general orientifold background.

By considering only Op^--planes with $p = 1 \ mod\ 4$, we shall propose in this section that Real KK-theory[5] is the correct group to classify Dd-branes in such backgrounds. Following closely [6], we shall show that our proposal can also reproduce some of the expected properties of non-BPS and BPS branes by studying the related Clifford Algebra.

[5] We adopt the convention in mathematical literature by refering to the orthogonal KK-theory as "real", and to the complex (with involution) one as "Real".

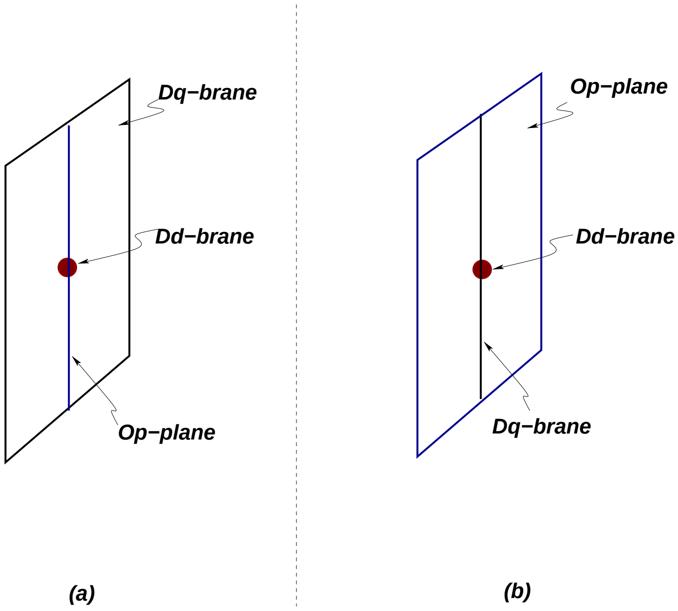

Figure 1: A Dd-brane constructed from a Dq-brane where $d < q$ and (a) Dq is dimensionally higher than the orientifold plane Op, (b) $q < p$.

3.1 Dd-branes from unstable Dq-branes in orientifold backgrounds and K-theory

In order to know how to construct KKR-theory groups, let us first construct a K-theory group which classifies Dd-branes on top of an orientifold plane. Here we do not consider the case in which (part of) the Dd-brane is constructed from lower dimensional D-branes. As far as we know, this group has not been reported in the literature. However, its construction is straightforward as we shall see.

The general situation can be divided in two diferent configurations: 1) The Op-plane is immersed in the unstable Dq-brane, i.e., $q \geqslant p$ and 2) the opposite situation in which $p \geqslant q$. We concentrate on those cases in which the Dd-brane is totally immersed in the orientifold plane. More general cases are taken into account in the KK-theory formalism.

- **Case 1:** $q \geqslant p$. The important issue is to construct the transversal space to the Dd-brane as depicted in Fig 1(a). It is easy to see that such space is given by $\mathbb{R}^{(9-q)+(q-p),p-d}$, from which we can construct the associated K-theory group as $KR(\mathbb{R}^{(9-q)+(q-p),p-d})$. By using the following relations for KR

$$\begin{aligned} KR(\mathbb{R}^{0,m}) &= KO(\mathbb{S}^m), \\ KR(\mathbb{R}^{n,m}) &= KR^{n,0}(\mathbb{R}^{0,m}) = KR^{0,m}(\mathbb{R}^{n,0}), \\ KR^{n,m}(X) &= KR(X \times \mathbb{R}^{n,m}), \\ KR^{n,m}(X) &= KR^{n-m}(X) = KR^{n-m\pm 8}(X), \end{aligned} \tag{3.1}$$

we can rewrite the K-theory group as

$$KR^{1-q}(\mathbb{R}^{q-p,p-d}), \tag{3.2}$$

where $\mathbb{R}^{q-p,p-d}$ is the transverse space of the Dd-brane respect the unstable Dq-brane system.

- **Case 2:** $p \geqslant q$. Let us now consider the depicted in figure 1(b). For the orientifold plane containing the unstable Dq-brane, the transversal space for the Dd-brane is $\mathbb{R}^{9-p,(p-q)+(q-d)}$, for which the corresponding K-theory group is

$$KR(\mathbb{R}^{9-p,p-d}) = KR^{9-2p+q}(\mathbb{R}^{0,q-d}), \tag{3.3}$$

where we have again used the isomorphisms for KR in the left hand side. Notice that the K-theory group written in such a way, allows us to identify the space $\mathbb{R}^{0,q-d}$ as the transversal one to the Dd-brane with respect to the Dq-brane, as in case 1.

Now, let us check if the above two formulae are consistent with what we already know. Essentially we have two limits to check. First of all, if $q = p = 9$ we reproduce immediately the known formula which classifies Dd-branes in Type I theory, i.e., $KO(\mathbb{R}^{9-d})$. The second limit to recover is Bergman's formula for Dd-branes in Type I theory, from unstable Dq-branes. Hence in this case, $p = 9$ but different from q. In such a case, the related K-theory group reads

$$KR^{q-1}(\mathbb{R}^{0,q-d}) = KO^{q-1}(\mathbb{R}^{q-d}), \tag{3.4}$$

which indeed validates our proposal.

3.2 The real KK-theory group

We now proceed to define the KKR-theory groups relatad to the configurations so far discused.

We start by introducing the formal definition for the Real KK-theory group which we shall apply in order to classify Dd-branes in orientifold backgrounds.

Real KK-theory groups are defined in terms of a Real C^*-algebra which is just a complex C^*-algebra with an additional antilinear involution $\mathcal{I}$ such that $\mathcal{I}(b_1 b_2) = \mathcal{I}(b_1)\mathcal{I}(b_2)$ and $\mathcal{I}(b^*) = (\mathcal{I}(b))^*$, for every b, b_1, b_2 in the complex C^*-algebra. Notice as well that (by definition) $\mathcal{I}(i) = -i$.

Now, let A and B be trivially graded, separable and unital Real C^*-algebras. An *even Kasparov Real A-B*-module is defined as for the complex and orthogonal cases (see appendices A and B for details and notation), with the following additional data:

- An antilinear Real involution $\mathcal{I}$ on $\mathcal{H}_B$ with the following property: $\mathcal{I}(xb) = \mathcal{I}(x)\mathcal{I}(b)$ and $(\mathcal{I}(x), \mathcal{I}(y)) = \mathcal{I}((x, y))$ for $x, y \in \mathcal{H}_B$ and $b \in B$.
- An antilinear Real involution $\mathcal{I}$ on $\mathbf{B}(\mathcal{H}_B) = M_2(M(B \otimes \mathcal{K}))$ defined by $\mathcal{I}(T)(x) = \mathcal{I}(T(\mathcal{I}(x)))$ for $x \in \mathcal{H}_B$.
- $\phi : A \to \mathbf{B}(\mathcal{H}_B)$ is a $*$-homomorphism of Real C^*-algebras, i.e. $\phi(\mathcal{I}(a)) = \mathcal{I}(\phi(a))$ for all $a \in A$.

The basic KK-group for Real C^*-algebras A, B will be denoted $KKR(A,B)$ and it is defined as the equivalence classes of even Kasparov Real $A-B$-modules with the equivalence relations defined as in the complex and real cases; with the additional requirement that both, the homomorphisms ϕ and the operators T appearing in the Kasparov modules; as well as the unitary operator generating the relation of unitary equivalence be *invariant* under the Real involution ($\mathcal{I}(a) = a$), i.e they belong to the *fixed point algebra* of the Real algebra to which they belong.

The corresponding higher KKR-groups are denoted as $KKR^{-n} \equiv KKR_n(A,B)$ and defined as in the real case, but using the Real Clifford algebras ${}^p\mathbb{C}^{n,m}$, with some Real involution $\mathcal{I}_p$ which is determined in our case by the orientifold Op^- action on the Clifford generators.[6] Hence, we denote the involution action on an element a of the Real Clifford algebra as $\mathcal{I}_{9-p}(a)$.

In this way, we have [17, 20]

$$KKR_{m-n+r-s}(A,B) = KKR(A \otimes {}^p\mathbb{C}l^{n,m}, B \otimes {}^p\mathbb{C}l^{r,s}). \tag{3.5}$$

The KKR^n-groups are periodic mod 8 and $KKR(A,B) = KKO(A,B)$ if both A and B have trivial Real involution [20]. A Bott periodicity result also holds for Real KK-theory:

$$\begin{aligned} KKR^k(X,Y) &= KKR^{k+m-n}(X \times \mathbb{R}^{m,n}, Y) = KKR^{k-m+n}(X, Y \times \mathbb{R}^{m,n}), \\ KKR^{-m}(pt,Y) &= KR^{-m}(Y). \end{aligned} \tag{3.6}$$

One important example that will be useful in section 4 is $Y = pt$. In terms of the Kasparov modules, the Real KK-theory group $KKR^{m-n}(C_0(X), pt) = KKR(C_0(X), {}^p\mathbb{C}l^{n,m})$ consist of equivalence classes of triples $({}^p\mathcal{H}, {}^p\phi, {}^pF)$ where ${}^p\mathcal{H} = {}^p\mathbb{C}^\infty \otimes {}^p\mathbb{C}l^{n,m}$ is the Hilbert space over $C_0(pt) \otimes {}^p\mathbb{C}l^{n,m} \approx \mathbb{C} \otimes {}^p\mathbb{C}l^{n,m}$, ${}^p\phi : {}^pC_0(X) \to {}^p\mathbf{B}({}^p\mathcal{H})$ is a $*$-homomorphism and ${}^pF$ is a self-adjoint operator in ${}^p\mathbf{B}({}^p\mathcal{H}) = {}^p\mathbf{B}({}^p\mathbb{C}^\infty) \otimes {}^p\mathbb{C}l^{n,m}$. On all of them, the index p means that there is an induced involution $\mathcal{I}_{9-p}$ (in our case from the orientifold action on the spacetime) with the properties mentioned above. Also we require for the tachyon and the scalar fields to be odd and even respectively under the $\mathbb{Z}_2$-grading.

In this context, the tachyon is written as

$$F = \sum_{A_l \in {}^p\mathbb{C}l^{n,m}_{\text{odd}}} T_l A_l, \tag{3.7}$$

where $T_l \in {}^p\mathbf{B}({}^p\mathbb{C}^\infty)$ which transforms on a representation determined by the self-duality condition $F = F^\dagger$ and the A_l form a basis for ${}^p\mathbb{C}l^{n,m}_{\text{odd}}$, which denotes the odd part of the Real Clifford algebra ${}^p\mathbb{C}l^{n,m}$ (see appendix C). Similarly, the unitary transformation $U \in {}^p\mathbf{B}({}^p\mathcal{H})$ on ${}^p\mathcal{H}$, which is a gauge transformation, is even with respect to the $\mathbb{Z}_2$-grading determined by $(-1)^{F_L}$. Hence, such transformation, together with the scalar fields, are written as

$${}^p\phi = \sum_{B_l \in {}^p\mathbb{C}l^{n,m}_{\text{even}}} \phi_l B_l, \tag{3.8}$$

[6] We denote the Real Clifford algebras as ${}^p\mathbb{C}^{p,q}$ in order to distinguish them from the complex Clifford algebras used in complex KK-theory. Also, we denote a generic field ψ under the action of the involution $\mathcal{I}$ determined by the orientifold p-plane as ${}^p\psi$.

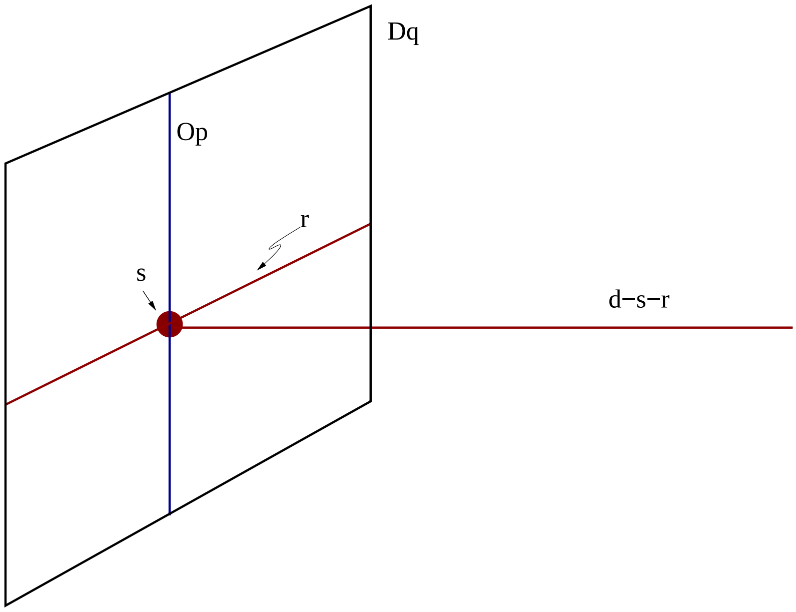

Figure 2: A Dd-brane constructed from tachyon condensation from unstable Dq-brane and unstable D-instantons. The orientifold Op lies inside the Dq-brane worldvolume.

where $\phi_l \in {}^p\mathbf{B}({}^p\mathbb{C}^\infty)$ and the corresponding representation is obtained from the condition ${}^p\phi = {}^p\phi^\dagger$. B_l form a basis for ${}^p\mathbb{C}l^{n,m}_{\text{even}}$, which denotes the even part of the Real Clifford algebra ${}^p\mathbb{C}l^{n,m}$.

Notice as well that the tachyon, scalar fields and the unitary transformation must be invariant under the orientifold action, i.e., written in terms of the Clifford algebra elements, they belong to the so-called fixed point algebra of the corresponding Clifford Algebra.[7] (See appendix C for details.)

3.3 D-branes in orientifolds and real KK-theory

With all the necessary ingredients we are in position to construct the relevant KKR-group which classifies Dd-brane in the presence of orientifold planes. As we have seen, one can construct it by analyzing the Dd-brane transversal space.

Let us start by identifying the spaces X and Y'. There are two different configurations according to the relative values between q and p, i.e., whether the plane Op^- is immersed in the unstable Dq-brane ($q > p$) or viceversa ($p > q$). Let us start with the first case as depicted in figure 2.

We fix our notation by claiming that the final Dd-brane is located in coordinates $x^0, \ldots, x^s, x^{p+1}, \ldots, x^{p+r}, x^{q+1}, \ldots, x^{q+d-s-r}$. Notice also that our assumption is that the subspace of the Dd-worldvolume of dimension $(s + r)$ is created by the usual tachyon

[7]It can be shown that an element of the fixed point algebra of ${}^p\mathbf{B}({}^p\mathbb{C}^\infty)$ is, roughly speaking an infinity real matrix with no involution; then the condition of belonging to the fixed point algebra of ${}^p\mathbf{B}({}^p\mathcal{H}) = {}^p\mathbf{B}({}^p\mathbb{C}^\infty) \otimes {}^p\mathbb{C}l^{n,m}$ is equivalent to belong to the fixed point algebra of ${}^p\mathbb{C}l^{n,m}$ times an infinity real matrix i.e we only need to know the fixed point algebra of ${}^p\mathbb{C}l^{n,m}$.

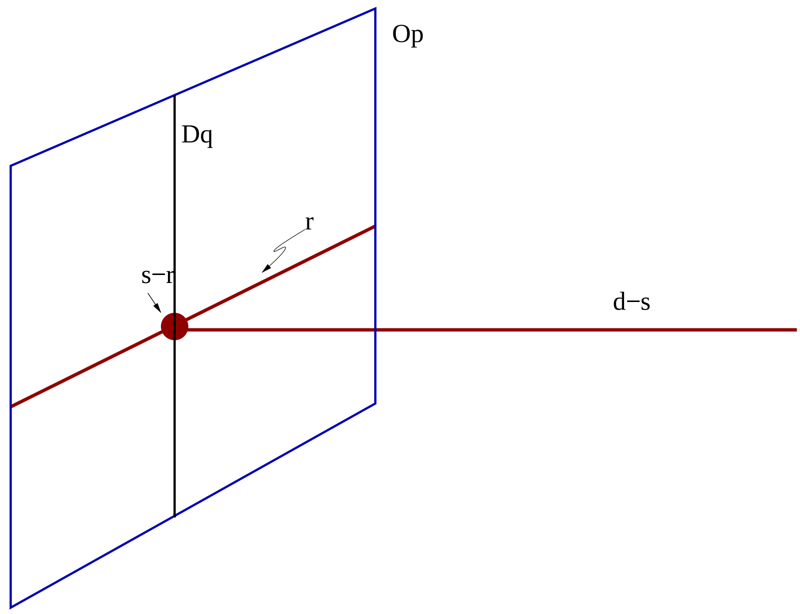

Figure 3: Dq-brane dimensionally lower than the orientifold plane Op.

condensation from the Dq-brane, while the subspace of dimension $(d-s-r)$ is gathered from tachyon condensation as in the K-matrix theory. Therefore, the transversal space Y' to the subspace of dimension $(r+s)$ is given by $\mathbb{R}^{(9-q)+(q-p-r),p-s}$, while the subspace X with dimension $(d-s-r)$ is $\mathbb{R}^{d-s-r,0}$. Hence it follows that the KKR-group classifying Dd-branes in this configuration is given by

$$KKR(\mathbb{R}^{d-s-r,0},\mathbb{R}^{(9-q)+(q-p-r),p-s}) = KKR^{1-q}(\mathbb{R}^{d-s-r,0},\mathbb{R}^{q-p-r,p-s}), \qquad (3.9)$$

where we have used the relations (3.6) for the last two terms. We can see that $Y = \mathbb{R}^{q-p-r,p-s}$.

Let us now focus in our second configuration, i.e., the case in which the Dq-brane is immersed in the orientifold plane Op^- as depicted in figure 3 $(p>q)$. Notice that in this case, there are some transversal coordinates of the Dd-brane with respect to the Dq-brane which are extended also inside the orientifold plane. We consider the Dd-brane to be extended in coordinates $x^0, x^1, \ldots, x^{s-r}, x^{q+1}, \ldots, x^{q+r}, x^{p+1}, \ldots, x^{p+d-s}$, while the unstable D$q$ and the orientifold are extended in coordinates labeled by their dimensions.

Hence the transversal space Y' is $\mathbb{R}^{9-p,p-q+(q+r-s)}$, while the space X is given by $\mathbb{R}^{d-s,r}$ such that the relevant KKR-group is

$$KKR(\mathbb{R}^{d-s,r},\mathbb{R}^{9-p,p-q+(q+r-s)}) = KKR^{9-2p+q}(\mathbb{R}^{d-s,r},\mathbb{R}^{0,q+r-s}), \qquad (3.10)$$

where in the last equality we have used the isomorphisms for KKR. Since we are working with $p=1,5,9$, this last group reduces to $KKR^{q-1}(\mathbb{R}^{d-s,r},\mathbb{R}^{0,q+r-s})$. Notice that in this way, we can identify the second entrance in the bifunctor $\mathbb{R}^{0,q+r-s}$ as the Dd-brane's transversal space within the Dq-brane.

There are actually two special limits we want to consider. In type I theory one has $p = 9$ and we should recover the results given in [6]. Indeed, in this case $(p > q)$ we get that $s = d$ and from (3.10)

$$\begin{aligned} KKR^{q-1}(\mathbb{R}^{d-s,r}, \mathbb{R}^{0,q+r-s}) &= KKR^{q-1}(pt, \mathbb{R}^{0,q-d}) \\ &= KR^{q-1}(\mathbb{R}^{0,q-d}) \\ &= KO^{q-1}(\mathbb{R}^{q-d}) \\ &= KKO^{q-1}(\mathbb{R}^{d+m-q}, \mathbb{R}^{m}), \end{aligned} \tag{3.11}$$

where $m = q - d + r$ is the codimension between the part of the Dd-brane inside of the unstable Dq-brane system.

The second limit we want to check is that of a Dd-brane located on top of an orientifold with $p \neq 9$ and for this we can take the case $q \geq p$. From figure 2 this configuration is equivalent to set $d = s$ and $r = 0$ in (3.9). Hence we have that $X = pt$ and

$$\begin{aligned} KKR^{1-q}(\mathbb{R}^{d-s-r,0}, \mathbb{R}^{q-p-r,p-s}) &= KKR^{1-q}(pt, \mathbb{R}^{q-p,p-d}) \\ &= KR^{1-q}(\mathbb{R}^{q-p,p-d}), \end{aligned} \tag{3.12}$$

which is in agreement with previous results from section 3.1.

One can ask what kind of extra information (with respect to K-theory) does we get from these groups. The main point (besides some more formal statements) is that we can have now a group which classifies D-branes intersecting orientifold planes. This can be achieved easily by noticing that eqs. (3.9) and (3.10) can be written as $KKR(\mathbb{R}^{d-s,0}, \mathbb{R}^{9-p,p-s})$ which satisfies

$$KKR(\mathbb{R}^{d-s,0}, \mathbb{R}^{9-p,p-s}) = KO(\mathbb{R}^{2p-2s+d-1}), \tag{3.13}$$

from which we can see that specific values of q and r are not important. This means that it does not matter which unstable brane we select to construct a Dd-brane, but how many coordinates s of the Dd-brane are inside the orientifold plane.

3.3.1 Example 1

It is easy to check that the orthogonal KO-group in (3.13) classifies D-branes in a Type I T-dual version. To see this, consider for instance a D3-brane in coordinates (x^0, x^1, x^2, x^3), and an orientifold $O1^-$ located in coordinates x^0, x^1. By applying T-duality on coordinates x^2-x^9, one gets a D7-brane in Type I theory. Such a brane is classified by $KO(\mathbb{R}^2) = \mathbb{Z}_2$. Now let us check if eq. (3.13) leads us to the same group. In this case, $q \geq p$ and the configuration is similar to that depicted in figure 2. It turns out that $s = p = 1$ and

1. $r = 2$ and $d - s - r = 0$, or
2. $r = 1$ and $d - s - r = 1$, or
3. $r = 0$ and $d - s - r = 2$.

For all cases, eq. (3.13) gives the group $KO(\mathbb{R}^2)$ in agreement with T-duality. The same applies for all different configuration of D-branes and orientifold planes. The KO-theory groups from eq. (3.13) classifies the T-dual version in Type I theory.

One can try to do the same for other type of orientifolds, like the ones with a negative square involution (positive RR charge) and for orientifolds in Type IIA. However, for such cases the related KK groups are not well known from the mathematical point of view. Hence, we can only establish some expected properties for such groups based on physical arguments. We shall comment on these issues in the section 5.

4. Unstable non-BPS D-branes in orientifolds and KKR-theory

We shall follow the criteria in [6] to show that eqs. (3.9) and (3.10) correctly classify Dd-branes in orientifold backgrounds. Hence, we shall extract the field content of unstable non-BPS Dq-branes from the Clifford algebra related to the KKR-group. As we have seen, the related Clifford algebra is a complex algebra with an antilinear involution induced by the orientifold action. In this section we shall obtain the Clifford algebra for each configuration of non-BPS Dq-branes and Op-planes, and we shall see that the field content perfectly agrees with that of an unstable non-BPS brane. Finally we use T-duality to show that the properties of the non-BPS branes are those expected from non-BPS branes in Type I theory.

Our proposal for the classification of Dd-branes in Op^--plane background is given by eq. (3.9) and eq. (3.10) according whether $q > p$ or$p > q$.

By using eq. (3.5) we have that

$$\begin{aligned} KKR^{1-q}(X,Y) &= KKR_{q-1}(X,Y) \\ &= \begin{cases} KKR(C_0(X;\mathbb{C}) \otimes {}^{p}\mathbb{C}l^{0,q-1}, C_0(Y;\mathbb{C}))\ q-1>0, \\ KKR(C_0(X;\mathbb{C}) \otimes {}^{p}\mathbb{C}l^{1-q,0}, C_0(Y;\mathbb{C}))\ 1-q>0, \end{cases} \end{aligned} \tag{4.1}$$

for $p < q$ while for $p > q$, we have

$$\begin{aligned} KKR^{q-1}(X,Y) &= KKR_{1-q}(X,Y) \\ &= \begin{cases} KKR(C_0(X;\mathbb{C}), C_0(Y;\mathbb{C}) \otimes {}^{p}\mathbb{C}l^{1-q,0})\ 1-q>0, \\ KKR(C_0(X;\mathbb{C}), C_0(Y;\mathbb{C}) \otimes {}^{p}\mathbb{C}l^{0,q-1})\ q-1>0. \end{cases} \end{aligned} \tag{4.2}$$

We shall use the above formulae as definition of the KKR-theory groups for D-brane classification. This choice is taken in order to recover the convention in [6] where $p = 9$ and consequently $q \leq p$. Since in this case the involution is trivial, (4.2) reduces to the definition for $KKO^{-n}(X,Y)$ used in [6]. In that sense, we classify non-BPS Dd-branes by making them to coincide with the unstable Dq-system, implying that $X = pt$ and that $Y = pt$. By this assumption we can safely conclude that all information about these non-BPS D-branes relies on the corresponding Clifford algebras. All what we need to specify is the involution action on the Clifford algebra generators.

4.1 The real involution and orientifolds

Let us describe explicity how the real involution acts on the Clifford generators induced by the orientifold Op^--plane.

	0	1	2	$\cdots$	q-1	q	q+1	$\cdots$	p-1	p	p+1	$\cdots$	8	9
Op^-	–	–	–	–	–	–	–	–	–	–	×	×	×	×
Dq	–	–	–	–	–	–	×	×	×	×	×	×	×	×

Table 1: Coordinates of the relative positions of a Dq-brane inside an orientifold plane Op ($q < p$).

The complexified Real Clifford algebra is defined as

$$ {}^{p}\mathbb{C}l^{n,m} = {}^{p}\left(Cl^{n,m} \otimes \mathbb{C}\right). \tag{4.3}$$

Since the involution acts as conjugation on the complex part we can write (see appendix C for details)

$$ {}^{p}\left(Cl^{n,m} \otimes \mathbb{C}\right) = {}^{p}Cl^{n,m} \otimes \overline{\mathbb{C}}, \tag{4.4}$$

where $\overline{\mathbb{C}}$ denotes the field of complex numbers with Real involution defined by usual complex conjugation and ${}^{p}Cl^{n,m}$ denotes the Clifford algebra $Cl^{n,m}$ with some Real involution (again, this involution is determined by the orientifold plane on the generators of the algebra and extended by linearity).Thus, it suffices to study the involution in the real part ${}^{p}Cl^{n,m}$. Hence, we shall concentrate on how to fix the involution inhereted from the orientifold Op^--plane on the generators of the real Clifford algebra.

According to eqs. (4.1) and (4.2) the complex Clifford algebras with involution we use, are of the form ${}^{p}\mathbb{C}l^{n,0}$ or ${}^{p}\mathbb{C}l^{0,n}$. Hence we shall concentrate on the involution on their associated orthogonal (real) Clifford algebras, whose generators can be identified with spatial coordinates via the vector space isomorphism

$$ Cl^{n,0} \cong \Lambda^{*}\mathbb{R}^{n} \cong Cl^{n,0}. \tag{4.5}$$

Let us consider the case in which $1-q < 0$ such that the related Clifford algebra is ${}^{p}\mathbb{C}l^{0,q-1}$. By the above isomorphism we identify the generators e_i, $(i = 1, \ldots, q-1)$ of the Clifford algebra $Cl^{0,q-1}$ with vectors of the little group SO$(q-1)$ of a Dq-brane.

Hence, the involution inhereted form the orientifold p-plane, denoted as $\mathcal{I}_{9-p}$ acts on the generators of the complex Clifford algebra as in the longitudinal coordinates x^i to the Dq-brane. Because of this, the involution depends on the relative value between p and q. Then, if $q < p$ we consider a Dq-brane inside the orientifold plane, as in table 1. This configuration induces a trivial involution on all the Clifford algebra generators

$$ \mathcal{I}_{9-p}(e_i) = e_i \quad \text{for all } i. \tag{4.6}$$

On the other hand, if $q > p$, the unstable Dq-brane is located as shown in table 2 and the involution is given by

$$ \mathcal{I}_{9-p}(e_i) = \begin{cases} e_i & \text{for} \quad i = 1, \ldots, p-1, \\ -e_i & \text{for} \quad i = p, \ldots, q+1. \end{cases} \tag{4.7}$$

	0	1	2	$\cdots$	p-1	p	p+1	$\cdots$	q-1	q	q+1	$\cdots$	8	9
Op^-	$-$	$-$	$-$	$-$	$-$	$-$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
Dq	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$\times$	$\times$	$\times$	$\times$

Table 2: A configuration consisting in an Op-plane inside an unstable Dq-brane with $q > p$.

By Bott periodicity and eqs. (4.1) and (4.2) we have

$$KKR(X,\ {}^p\mathbb{C}l^{0,q-1}) = KKR(X,\ {}^p\mathbb{C}l^{9-q,0}). \tag{4.8}$$

In this way, we can use instead the $(9-q)$ generators e_i of ${}^p\mathbb{C}l^{9-q,0}$, with $i = q+1, \ldots, 9$ which are identified with the transversal coordinates to the Dq-brane. The involution is again dependent on the relative values between q and p. For $q < p$ (see figure 3) we have

$$\mathcal{I}_{9-p}(e_i) = \begin{cases} \ \ e_i & \text{for} \quad i = 1, \ldots, p-q, \\ -e_i & \text{for} \quad i = p-q+1, \ldots, 9-q, \end{cases} \tag{4.9}$$

while for $q > p$ we have

$$\mathcal{I}_{9-p}(e_i) = -e_i \qquad \text{for all } i. \tag{4.10}$$

Therefore, one sees that for $q > 2$, we have at least two different ways to identity the Clifford algebra generators with spatial coordinates i.e. internal or transversal coordinates to the Dq-brane system. For each identification there are two choices for the involution on the Clifford generators, depending on the relative value of q and p. However we also see that for $q < p$ is simpler to establish the identification with internal coordinates to the Dq-brane, while for the case $p < q$ is the opposite. We shall adopt this identification henceforth.

Although the identifications are not so geometric for $q < 2$, we have similar involutions. For $q = -1$ the relevant Clifford algebra is ${}^pCl^{2,0}$ and the involution acts on the generators as $\mathcal{I}_{9-p}(e_i) = e_i$ $(i = 1, 2)$. Similarly for $q = 0$, the Clifford algebra is ${}^pCl^{1,0}$ and the involution acts also trivially on the generator.

4.2 Non-BPS D-branes in orientifold backgrounds

Now, we are going to get the representations of the tachyon, gauge and scalar fields from the corresponding Clifford algebras, following the procedure used in [6], and we will show that they correspond to the properties of unstable non-BPS Dd-branes classified by the groups in eqs. (3.9) and (3.10). Due to Bott periodicity in $KKR^n(X,Y) \sim KKR^{n\pm 8}(X,Y)$, all cases are considered within the range $-4 \leq n \leq 4$. However, in contrast with D-branes in Type I theory, the involution acts different for a Dq-brane than for a D$(q+8)$-brane. Notice as well that, although eqs. (3.9) and (3.10) do not depend on p (in these kinds of non-BPS branes), the involution does.

	0	1	2	3	4	5	6	7	8	9
$O1^-$	−	−	×	×	×	×	×	×	×	×
$D8$	−	−	−	−	−	−	−	−	−	×.

Table 3: Relative positions of the $O1^-$-plane and the D8-brane from example 1.

4.2.1 Example 1

Consider for instance the case of a non-BPS D8-brane and an $O1^-$-plane in a configuration as described in table 3.

The corresponding group is $KKR^{-7}(pt, pt) \sim KKR^1(pt, pt)$ with an associated Real Clifford algebra ${}^1\mathbb{C}l^{1,0}$. The action of the involution on the generator of ${}^1Cl^{1,0}$ is given by

$$\mathcal{I}_8(e_1) = -e_1. \tag{4.11}$$

This determines the fixed point algebra for ${}^1\mathbb{C}l^{1,0}$ and hence, the corresponding representation for the tachyon, gauge and scalar fields. By imposing the condition $\mathcal{I}_8(a) = a$ for $a \in {}^1\mathbb{C}l^{1,0}$, one gets that

$$\left({}^1\mathbb{C}l^{1,0} \right)_{\text{fix}} = Cl^{0,1}, \tag{4.12}$$

which fixes the tachyon T and the scalar field ϕ to be symmetric tensor representations $\square\!\square$ of the gauge group $O(\infty)$. As it was shown in [6], these results correspond to the field content of an unstable non-BPS D2-brane in Type I theory. This is in agreement with formula (3.9) since for this case[8] $p = s = 1$, $d = q = 8$ and $r = 7$, and the relevant KKR group is given by

$$KKR^{-7}(pt, pt) = KO(\mathbb{R}^7) = 0, \tag{4.13}$$

which indeed is the K-theory group which classifies D2-branes in Type I theory. One can as well check that under T-duality on transversal coordinates to the $O1^-$-plane, the unstable D8-brane transforms into a D2-brane in Type I theory. Notice that the involution does not change for $p = 5$, for which we get the same field content for a D8 in an $O5^-$-plane.

4.2.2 Example 2

Contrary to the case in Type I theory, the field content for a non-BPS D0-brane in an $O1^-$-plane (as shown in table 4) should not be the same than for a D8. This is obtained by realizing that for a D0-brane, although the Real Clifford algebra also is ${}^1\mathbb{C}l^{1,0}$, the involution on the single one generator e_1 is trivial, $\mathcal{I}_8(e_1) = e_1$. This implies that

$$\left({}^1\mathbb{C}l^{1,0} \right)_{\text{fix}} = Cl^{1,0}. \tag{4.14}$$

Therefore, the tachyon field T and the scalar field ϕ are antisymmetric $\begin{smallmatrix}\square\\[-4pt]\square\end{smallmatrix}$ and symmetric $\square\!\square$ tensor representations, respectively, of the gauge group $O(\infty)$ [6]. This field content is

[8] Actually, as we shall see, similar conditions hold for all unstable non-BPS D-branes.

	0	1	2	3	4	5	6	7	8	9
$O1^-$	–	–	×	×	×	×	×	×	×	×
$D0$	–	×	×	×	×	×	×	×	×	×

Table 4: Relative positions of a D0-brane inside an $O1^-$-plane as described in example 3.

precisely that of an unstable non-BPS D0-brane in Type I theory. This also is in agreement with formula (3.10) in which $r = s = q = d = 0$ and $p = 1$, implying

$$KKR^{-1}(pt, pt) = KO(\mathbb{R}^1) = \mathbb{Z}_2, \tag{4.15}$$

which classifies D8-branes in Type I theory. Indeed, the configuration of a D0-brane in an $O1^-$-plane is T-dual to a D8 in an $O9^-$-plane. For $p = 5$, the involution is the same and we get the same group.

4.2.3 Example 3

Another interesting situation presents for $q = 5$, i.e., D5-branes in $O1^-$ and $O5^-$-planes. The Real Clifford algebra is given by ${}^p\mathbb{C}l^{4,0}$ for $p = 1, 5$. In this case, the involution acts as $\mathcal{I}_4(e_i) = -e_i$ for $i = 2, 3, 4, 5$. As a consequence, the fixed point algebra is $Cl^{0,4}$. For this case, we can also take the Real Clifford algebra as ${}^p\mathbb{C}l^{4,0} = {}^p\mathbb{C}l^{0,4}$. However the involution acts trivially on the corresponding generators. The fixed point algebra is then $Cl^{4,0}$. It is easy to check that $Cl^{4,0} = Cl^{0,4}$. Hence, as it was shown in [6], the tachyon and scalars fields transforms in the bifundamental and antisymmetric tensor representations of the gauge group $\mathrm{Sp}(\infty) \times \mathrm{Sp}(\infty)$. This is the field content of a pair D5-$\overline{\text{D5}}$ branes in Type I theory, which agrees with the result given by

$$KKR^{-4}(pt, pt) = KO(\mathbb{R}^4) = \mathrm{KSp}(pt) = \mathbb{Z}. \tag{4.16}$$

The complete set of Real Clifford algebras for all unstable non-BPS branes is summarized in table 5. The representations and gauge groups for each case are recovered from the results shown in [6] just by computing the fixed point algebras, as in the previous examples. For completness we summarize such results in appendix C (see table 7).

4.3 Dq-branes from D-instantons in orientifold backgrounds

As we have said, we shall follow the criteria in [6] to test the validity of formulae (3.9), (3.10). For that we are going to show explicitly the construction of a Dd-brane from an infinitely many number of instantons in the presence of an orientifold plane $O1^-$ or $O5^-$. In [6] the authors found that the tension of Dd-branes in Type I theory are related to the size (dimension of the representation) of SO(d) gamma matrices. In the case of lower dimensional orientifold planes, we shall get a similar relation.

The strategy in [6] adapted to our case is as follows. An explicit configuration representing a Dd-brane is gathered by constructing the corresponding configuration in Type IIB, based on D-instanton-anti-D-instanton, which survives after the orientifold projection.

JHEP12(2008)007

	Dd	${}^p\mathbb{C}l^{n,m}$	$({}^pCl^n)_{\text{fix}}$	KKR^n	$KO^n(pt)$	T-dual in Type I
$p=1$ $p=5$	D(-1)	${}^1\mathbb{C}l^{2,0}$ ${}^5\mathbb{C}l^{2,0}$	$Cl^{2,0}$	KKR^{-2} KKR^{-10}	$KO^{-2}=\mathbb{Z}_2$ $KO^{-10}=\mathbb{Z}_2$	D7 D(-1)
	D0	${}^1\mathbb{C}l^{1,0}$ ${}^5\mathbb{C}l^{1,0}$	$Cl^{1,0}$	KKR^{-1} KKR^{-9}	$KO^{-1}=\mathbb{Z}_2$ $KO^{-9}=\mathbb{Z}_2$	D8 D0
	D1-$\overline{\text{D1}}$	${}^1\mathbb{C}l^{1,1}$ ${}^5\mathbb{C}l^{1,1}$	$Cl^{1,1}$	KKR^{0} KKR^{-8}	$KO^{0}=\mathbb{Z}$ $KO^{-8}=\mathbb{Z}$	D9-$\overline{\text{D9}}$ D1-$\overline{\text{D1}}$
	D2	${}^1\mathbb{C}l^{0,1}$ ${}^5\mathbb{C}l^{0,1}$	$Cl^{1,0}$ $Cl^{0,1}$	KKR^{-1} KKR^{-7}	$KO^{-1}=\mathbb{Z}_2$ $KO^{-7}=0$	D8 D2
	D3	${}^1\mathbb{C}l^{0,2}$ ${}^5\mathbb{C}l^{0,2}$	$Cl^{2,0}$ $Cl^{0,2}$	KKR^{-2} KKR^{-6}	$KO^{-2}=\mathbb{Z}_2$ $KO^{-6}=0$	D7 D3
	D4	${}^1\mathbb{C}l^{0,3}$ ${}^5\mathbb{C}l^{0,3}$	$Cl^{3,0}$ $Cl^{0,3}$	KKR^{-3} KKR^{-5}	$KO^{-3}=0$ $KO^{-5}=0$	D6 D4
	D5-$\overline{\text{D5}}$	${}^1\mathbb{C}l^{0,4}$ ${}^5\mathbb{C}l^{0,4}$	$Cl^{4,0}$	KKR^{-4}	$KO^{-4}=\mathbb{Z}$	D5+$\overline{\text{D5}}$
	D6	${}^1\mathbb{C}l^{3,0}$ ${}^5\mathbb{C}l^{3,0}$	$Cl^{0,3}$	KKR^{-5}	$KO^{-5}=0$	D4
	D7	${}^1\mathbb{C}l^{2,0}$ ${}^5\mathbb{C}l^{2,0}$	$Cl^{0,2}$	KKR^{-6}	$KO^{-6}=0$	D3
	D8	${}^1\mathbb{C}l^{1,0}$ ${}^5\mathbb{C}l^{1,0}$	$Cl^{0,1}$	KKR^{-7}	$KO^{-7}=0$	D2

Table 5: KKR-groups and their related Clifford algebras, fixed point algebras and KO-theory groups for unstable Dq-branes in $O1^-$ and $O5^-$-planes. The empty entries stand for the same expressions as the preceding row.

Hence, since the relevant Real Clifford algebra related to a system of D(-1)-$\overline{\text{D}(-1)}$ is $\mathbb{C}l^{2,0}$, and being the tachyon field odd with respect to the $\mathbb{Z}_2$-grading, it can be written as

$$F = T_1\widehat{e}_1 + T_2\widehat{e}_2, \tag{4.17}$$

where $\widehat{e}_1, \widehat{e}_2 \in \mathbb{C}l^{2,0}_{\text{odd}} = (Cl^{2,0}_{\text{odd}} \otimes \mathbb{C})$, and T_1 and T_2 are real fields. Besides this, the tachyon

JHEP12(2008)007

field is self-dual ($F = F^\dagger$) and is invariant under the involution $\mathcal{I}_{9-p}$, i.e.

$$\mathcal{I}_{9-p}(F) = F, \tag{4.18}$$

which makes it belongs to the fixed point algebra of the corresponding Real Clifford algebra. Then, the tachyon field can also be written as

$$F = T_a e_1 + T_b e_2, \tag{4.19}$$

where T_a and T_b are complex fields and $e_1, e_2 \in Cl^{2,0}_{\text{odd}}$. Defining the field $T = T_a + T_b e_1 \wedge e_2$ one gets that $T = -T^\dagger$ due to the self-duality condition on F. In particular, we observe that for an $O9^-$-plane, the involution acts trivially on all Clifford algebra generators. This implies that Im $T_a =$ Im $T_b = 0$ and that $T = -T^{\mathrm{T}}$.

Now, since for a Dd-brane constructed from instantons, the tachyon field also reads

$$F = \mu \sum_{i=0}^{d} p_i \otimes \Gamma^i, \tag{4.20}$$

comparing with eq. (4.19) we conclude that

$$\begin{aligned} T_a &= \partial_0 \otimes \gamma^0, \\ T_b &= \partial_j \otimes \gamma^j_d, \end{aligned} \tag{4.21}$$

with $(\gamma^\mu_{d+1})^\dagger = \gamma^\mu_{d+1}$ being hermitian γ-matrices, which in the abscence of an orientifold plane, are irreducible hermitian SO$(d+1)$ gamma matrices. In the presence of orientifold planes $O5^-$ and $O1^-$, it turns out that the involutions $\mathcal{I}_4$ and $\mathcal{I}_8$ act trivially on the generators e_1 and e_2 (as in the Type I case). This renders the gamma-matrices to split into $\gamma^0 = I$ and SO(d) gamma matrices γ^i ($i = 1, \ldots, d$) with the latter forming a real representation of $Cl^{0,d}$. Using this information we can compare the size of the tachyon in Type IIB and in the presence of orientifold planes. The ratio does not depend on p, implying that the tension (and size) of a Dd-brane in an $O5^-$, $O1^-$ and $O9^-$ (as in the configurations considered in the previous section) is twice than that in Type IIB for $d = 3, 4, 5, 6, 7$. Notice that for $p = 1, 5$ the Dd-branes with twice the tension than in Type IIB are T-duals to those in Type I theory which also have twice the tension as their counterparts in Type IIB. This is shown in table 6. Notice as well, as it was pointed out in [6], that this is consistent with the construction of D-branes in Type I theory, since those branes in an Op^--plane with twice the tension than in Type IIB, are T-duals to Type I D-branes constructed from two Type IIB branes or a pair of brane-antibrane.

This is our last test to show that indeed, KKR-theory truly classifies D-branes charges in (the provided) orientifold backgrounds.

5. A proposal for classification of D-branes in Op^+-planes

In [1] and [6] KK-theory and KKO-theory are used to classify D-branes in Type II and Type I superstring theories respectively. Also in this paper we have extended this classification to orientifold backgrounds in Type IIB string theory by using KKR-theory.

JHEP12(2008)007

Dd	D0	D1	D2	D3	D4	D5	D6	D7	D8	D9
Size in IIB	1	1	2	2	4	4	8	8	16	16
Size in IIB + Op-plane, $p = 1, 5$	1	1	2	4	8	8	16	16	16	16
T-dual into Type I ($p = 1$)	D8	D7	D8	D7	D6	D5	D4	D3	D2	D1
T-dual into Type I ($p = 5$)	D0	D1	D2	D3	D4	D5	D4	D3	D2	D1

Table 6: Relative dimension of the representation between gamma matrices related to Dd-branes in Type IIB and in Op^--backgrounds with $p = 1, 5$.

Then, it is natural to think about the possibility of other KK-theories,[9] extending the K-theory classification of superstring theories in different backgrounds than those appearing in this paper. In particular, we focus on Type IIB Op^+ orientifolds (the involution induced on the Chan-Paton bundles is $\tau^2 = -1$) which are classified by quaternionic K-theory, denoted KH [13] and symplectic USp(32) (IIB + O9$^+$) string theory proposed in [21] which is classified by symplectic K-theory, denoted KSp.

We focus on these particular backgrounds because their associated K-theories have close relation with KO and KR theories[10] and consequently, we can conjecture some relations that their corresponding KKH and KKSp theories must satisfy.

For this purpose, we first write some properties and relations between KH, KSp and KO theories:

$$KH(X) \simeq KH^{-8}(X), \tag{5.1}$$

$$KH^{p,q}(X) \simeq KH^{p+1,q+1}(X) \simeq KH^{p-q}(X), \tag{5.2}$$

$$KH(X_R) \simeq \mathrm{KSp}(X_R), \tag{5.3}$$

$$\mathrm{KSp}(S^n) \simeq KO(S^{n+4}). \tag{5.4}$$

In (5.3), X_R is the fixed point set of the involution of the spacetime and this property reflects the fact that KH-theories are T-duals of USp(32) theory, with the involution acting on the dualized coordinates. For example if we start with USp(32) theory and we do not make any T-duality, then the involution does not act at all in the spacetime; so in this case the fixed spacetime is the fixed point set of the involution; in this way $\mathrm{KSp}(X) = KH(X)$ and USp(32) theory can be regarded as (IIB + O9$^+$)-string theory, in the same way Type I string theory can be seen as (IIB + O9$^-$)-string theory.

The most important property we shall assume in all KK-theory groups $KK^{-n}(X, Y)$ proposed here is that when either X or Y is the one point space, they reduce to the respective K-theory and K-(analytic) homology functors.[11] One consequence of this prop-

[9]At least those KK-theories related with K-theories classifying consistent stringy backgrounds.

[10]In [6], though they do not make explicit mention of KKSp theory, they use the relation between KO and KSp theories to conclude that USp(32) theory is classified by KKO^{q+3}, where q is de dimension of the unstable D-brane in the USp(32) theory. But USp(32) string theory is a consistent theory, then there should exist KKSp theory, which should be related to KKO-theory in a suitable way to achieve the KKO-groups proposed in [6].

[11]As in the case of KR-homology, there should be a suitable definition of topological K-homology and it must be possible to prove the equivalence with the analytical K-homology defined above.

JHEP12(2008)007

erty is that our KK-functors must preserve the original periodicity of their K-functors, i.e $n\ mod\ 8$ periodicity.

Let us start with (IIB + $O9^+$) backgrounds, i.e. KKH-theory. Both the crucial formula (3.6) of KKR-theory and the similar property (5.2) of KH-theory shared by KR-theory allowed us to compute the KKR-groups and to confirm our proposal; then we also assume that KKH-theory should obey a similar property:

$$KKH^k(X,Y) = KKH^{k+p-q}(X \times \mathbb{R}^{p,q}, Y) = KKH^{k-p+q}(X, Y \times \mathbb{R}^{p,q}). \tag{5.5}$$

Suppose we have a configuration similar to that of figure 2 (for our present purposes it is enough to restrict our attention to this system; but it is straightforward to adapt the following arguments for the configuration of figure 3. In analogy with the Op^- orientifolds, we propose that the KKH-group classifying stable D-brane configurations is given by

$$KKH(\mathbb{R}^{d-s-r,0}, \mathbb{R}^{(9-q)+(q-p-r),p-s}). \tag{5.6}$$

In this way the calculations are identical to the ones that lead to (3.9); so we have

$$KKH(\mathbb{R}^{d-s-r,0}, \mathbb{R}^{(9-q)+(q-p-r),p-s}) = KKH^{1-q}(\mathbb{R}^{d-s-r,0}, \mathbb{R}^{q-p-r,p-s}), \tag{5.7}$$

which can be written, by using (5.1)–(5.4), in the following way:

$$KKH^{1-q}(\mathbb{R}^{d-s-r,0}, \mathbb{R}^{q-p-r,p-s}) = \mathrm{KSp}(\mathbb{R}^{2p-2s+d-1}) = KO(\mathbb{R}^{2p-2s+d+3}). \tag{5.8}$$

If we take $r = 0$ and $d = s$, then the stable Dd-brane is located on top of the orientifold plane and (5.8) reduces to:

$$KKH^{1-q}(\mathbb{R}^{0,0}, \mathbb{R}^{q-p,p-d}) = \mathrm{KSp}(S^{2p-d-1}) = KO(S^{2p-d+3}), \tag{5.9}$$

which is precisely Gukov's prescription for D-branes located on top of Op^+ orientifolds. Then the basic properties of KKH-groups mentioned above are enough to carry on the classification of stable D-branes in Op^+ orientifolds.

To construct the corresponding *"quaternionic Kasparov module"* the first step is to define a *"quaternionic C^*-algebra"*; which means a Banach $*$-algebra A over the quaternionic field such that, the C^*-equation $\|x^*x\| = \|x\|^2$ holds for any $x \in A$.

Then, one can follows the path traced in [17] by substituting the fields $\mathbb{R}$ or $\mathbb{C}$ by $\mathbb{H}$; and the complex, real and Real Clifford algebras by the quaternionic Clifford algebras $\mathbb{C}l_H^{n,m}$[12] endowed with some C^*-algebra structure. Of course, along the way there may be some subtleties associated with the specific properties of $\mathbb{H}$, such as noncommutativity.

Now, we turn to USp(32) string theory. Suppose that in this theory we have a configuration similar to the one described in the paragraph above equation (2.15). Then we postulate (in accordance with (2.17)) that stable D-branes are classified by

$$KKSp^{q-1}(\mathbb{R}^{d+s-q}, \mathbb{R}^s). \tag{5.10}$$

[12] $\mathbb{C}l_H^{n,m}$ is defined as the tensor product of the real Clifford Algebra $Cl^{m,n}$ with the quaternionic field, i.e $\mathbb{C}l_H^{n,m} = Cl^{m,n} \otimes \mathbb{H}$.

In order for (5.10) to reproduce the K-theory group of the transverse space of the Dd-brane, we postulate the following property analogous to (B.5):

$$KKSp^k(X,Y) = KKSp^{k-n}(X \times \mathbb{R}^n, Y) = KKSp^{k+m}(X, Y \times \mathbb{R}^m). \tag{5.11}$$

In this way we get

$$\begin{aligned} KKSp^{q-1}(\mathbb{R}^{d+s-q}, \mathbb{R}^s) &= \mathrm{KSp}(\mathbb{R}^{9-p}) \\ &= KO^{-4}(\mathbb{R}^{9-p}) = KKO^{q+3}(\mathbb{R}^{d+s-q}, \mathbb{R}^s). \end{aligned} \tag{5.12}$$

From the above equation we reproduce the claim in [6] that D-branes in USp(32) string theory are classified by $KKO^{q+3}(X,Y)$. So, we claim that

$$KKSp^i(X,Y) = KKO^{i+4}(X,Y) = KKO^{i-4}(X,Y). \tag{5.13}$$

5.1 An application: exotic orientifolds

We know that for $p < 6$ there are a variatey of orientifold planes, characterized by their RR and NS-NS charge [22 − 24]. It is interesting to realize that a cohomological classification of the RR and NS fluxes, tells us that there are at least 4 different types of orientifold planes for $p < 6$ but only 3 in a K-theoretical classification [25, 24].[13]

At the level of cohomology, there are two different types of orientifold related to RR fluxes. They are classified by the torsion part of the group $H^{6-p}(\mathbb{RP}^{8-p}, \widetilde{\mathbb{Z}}) = \mathbb{Z}_2$ which is interpreted as a half-shift in RR charge, defining the exotic orientifold planes $\widetilde{Op}$. The brane realization of this type of orientifold plane Op is depicted in figure 4, where roughly speaking, an exotic $\widetilde{Op}$-plane is constructed by wrapping a D$(p+2)$-brane in a two-cycle of the transverse space $\mathbb{RP}^{8-p}$ of an Op-plane .

However, it can be shown [24] that a K-theoretical classification of RR fields gives more information such as an explanation for the relative charge between different types of orientifold planes. In this context Op^- and Op^+-planes are classified (through their RR fields) by $KR^{p-10}(\mathbb{S}^{9-p,0})$ and $KR^{p-6}(\mathbb{S}^{9-p,0}) = KH^{p-10}(\mathbb{S}^{9-p})$ respectively. For $p = 1, 5$ we have the values

$$\begin{aligned} O1^- &: \quad KR^{-1}(\mathbb{R}^{8,0}) = \mathbb{Z} \oplus \mathbb{Z}_2, \\ O1^+ &: \quad KR^{-5}(\mathbb{R}^{8,0}) = \mathbb{Z}, \\ O5^- &: \quad KR^{-5}(\mathbb{R}^{4,0}) = \mathbb{Z}, \\ O5^+ &: \quad KR^{-1}(\mathbb{R}^{4,0}) = \mathbb{Z} \oplus \mathbb{Z}_2. \end{aligned} \tag{5.14}$$

The abscence of a torsion part in the group for $O5^-$-planes, is interpreted (via the Atiyah Hirzebruch Spectral Sequence) as a shift in the RR charge by a half-unit, explaining the relative fractional charge between them and the exotic ones denoted $\widetilde{O5^-}$. In this sense is easy to see that $\widetilde{O5^+}$ has the same RR charge than $O5^+$. For the case of $O1$-planes, we have

[13] Actually, if one consider an S-dual version of the conecction between cohomology and K-theory (called the Atiyah-Hirzebruch Spectral Sequence) there are just two different types of orientifolds classified by K-theory [26, 27].

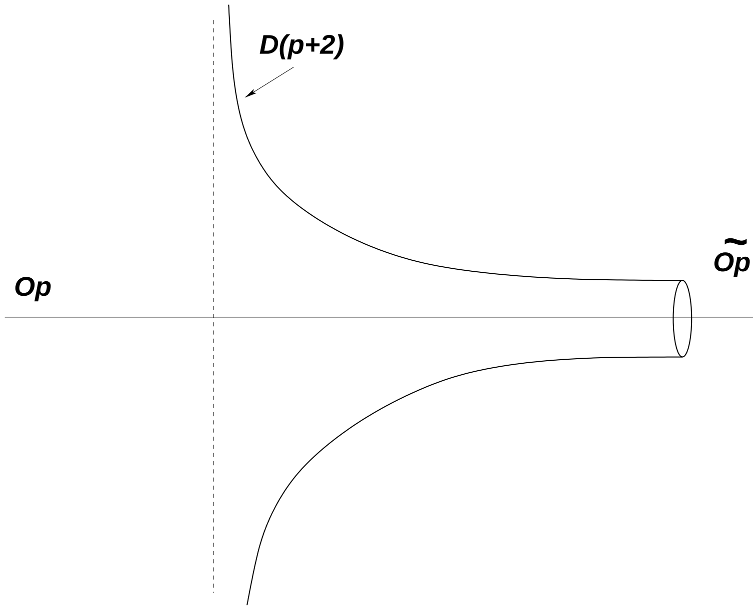

Figure 4: Brane realization of $\widetilde{Op}$-planes.

exactly the same situation, although there are extra $\mathbb{Z}_2$ constributions from cohomology. Once we compare this information with the corresponding K-theory groups, we arrive to the same conclusions as for the $O5$-planes [24].

Hence, although the existence of these exotic orientifolds comes from cohomology and a more accurate description about their RR charges is given by K-theory, this is actually a classification of RR fields. The D-brane realization of exotic orientifolds suggests on the other hand, a K-theory classification of D-brane (charges). Since an $\widetilde{Op}$-plane is constructed by a D($p+2$)-brane wrapping a two-cycle transversal to the orientifold, it should be enough to classify such configuration of branes in an orientifold background to elucidate their existence. This is precisely what KKR-theory does at least for $p = 1, 5, 9$.

Consider for instance the case of an $O5^-$-plane and a D7-brane wrapping a two-cycle transversal to the orientifold plane. Let us take the configuration given by $q < p$ (the same result can be obtained by taking $p < q$). Hence we have $d - s = 2$ and $s = 5$. The related KKR-theory group is

$$KKR^{-6}(pt, pt) = KO(\mathbb{R}^6) = 0, \tag{5.15}$$

which tells us that there is no extra contribution in K-theory to the $O5^-$-planes. On the other hand, for the $O1$-planes, we have that $d - s = 2$ and $s = 1$. The related KKR-theory group is then

$$KKR^{-2}(pt, pt) = KO(\mathbb{R}^2) = \mathbb{Z}_2, \tag{5.16}$$

which is also in agreement with (5.14). Finally, we can check that for the cases of $O5^+$ and $O1^+$, the proposed KKH-theory groups gives the expected results. For the $O5^+$-plane we have that the relevant KK-theory group is

$$KKH^{-6}(pt, pt) = \mathrm{KSp}(\mathbb{R}^6) = KO(\mathbb{R}^2) = \mathbb{Z}_2, \tag{5.17}$$

while for $O1^+$ we have that

$$KKH^{-2}(pt, pt) = \mathrm{KSp}(\mathbb{R}^2) = KO(\mathbb{R}^6) = 0. \tag{5.18}$$

This confirms that an orientifold classification in terms of branes rather than fields is easily gathered by KK-theory.

6. D-branes in orbifold singularities and KK-theory

So far our main focus has been on the prescription of D-branes in orientifolds. In this section we describe how to incorporate the equivariant version $KK_G(X, Y)$ of the Kasparov KK-theory bifunctor to the Dd-brane classification scheme. The expected group is $KK_G(X, Y)$ since the K-theory group classifying D-branes in orbifold singularities is the equivariant group $K_G(X)$ [10, 28].

For simplicity we will concentrate in the case where the dimension q of the unstable Dq-brane system is higher than the dimension p of the transverse space to the orbifold singularity. The reader can extend the formulation for $q \leq p$ by following the arguments in this section and section 3.3.

6.1 Type IIB orbifolds and equivariant KK_G-theory

In order to describe orbifold singularities with equivariant KK_G-theory we assume a group G acting on the $(9-p)$ coordinates $(x^{p+1}, \ldots, x^9)$ of spacetime in a Type IIB string theory, i.e. the spacatime is

$$\mathbb{R}^{p+1} \times (\mathcal{M}^{9-p}/G). \tag{6.1}$$

Let us concentrate on the case of flat spacetime by taking: $\mathcal{M}^{9-p} = \mathbb{R}^{9-p}$. Then, the general form of a subspace of spacetime is as follows:

$$\mathbb{R}^{\alpha,\beta} = (\mathbb{R}^\alpha/G) \times \mathbb{R}^\beta. \tag{6.2}$$

In [29] it is shown that the K-theory group classifying D-branes in orbifold singularities in Type IIB string theory is the equivariant group $K_G(X)$, where X is the transverse space to the Dd-brane with respect to the spacetime (or an unstable system of D9-$\overline{\text{D9}}$ spacetime filling D-branes). The arguments explained in section 2.1 for the IIB string theory can be applied here. Indeed, the K_G-group classifying stable D-branes in the Type IIB orbifold singularity with respect to an unstable Dq-brane system is given by $\mathrm{K}_G^{q-1}(X)$, where X is now the transverse space of the stable Dd-brane relative to the unstable Dq-brane system, with $p \leq q$.

Our goal is to classify all possible stable Dd-branes in the spacetime (6.1) by using equivariant KK_G-groups and incorporating the unstable information of the Dq-brane system mentioned above.

Using the above remarks and appendix B.1, we claim that the group classifying any Dd-brane located in the orbifold singularity is $KK_G^{q-1}(X,Y)$, where Y is the portion of the spacetime supporting the unstable Dq-branes and X is the transverse space to Y with coordinates $(x^{q+1},\ldots,x^9)$ in the whole spacetime. The present subsection will be devoted to prove this claim.

It is worth to mention again the limiting cases. For Dd-branes extended totally outside the worldvolume of the unstable system, we can take Y as a point. Then $KK_G^{q-1}(X,pt)$ is the group classifying D-branes extended along X. This is precisely the K$_G$-homology of X which classifies D-branes by their worldvolume. Similarly, if the D-brane is extended completely inside the unstable system, the group classifying stable D-branes is the equivariant K-group $KK_G^{q-1}(pt,Y) = K_G^{q-1}(Y)$ classifying in terms of the transverse space of the D-brane relative to the unstable system.

Thus in the general case, given a Dd-brane whose position lies both inside and outside of the unstable Dq-brane ambient, the two entries of the KK$_G$-functor should be filled firstly by the worldvolume X of the portion of the D-brane outside the unstable system. The second item Y of codimension m corresponds to the transverse space of the Dd-brane in the unstable Dq-brane.

Then the spaces filling the KK$_G$-functor entries depend strongly in the directions where the Dd-brane is extended, but not on its dimension d.

To be more specific, consider an unstable system of Dq-branes placed at the orbifold singularity and extended along $(x^0,\ldots,x^q)$ and place a Dd-brane extended along $(x^0,\ldots,x^{q-m},x^{q+1},\ldots,x^{d+m})$ where the spacetime is the orbifold defined above,[14] with $q-m \leqq p \leqq q$. Then the KK$_G$-theory group classifying this system is

$$KK_G^{q-1}(\mathbb{R}^{d+m-q,0},\mathbb{R}^{q-p,p-q+m}). \tag{6.3}$$

Using (D.1) and assuming that G acts by a spinor representation we find that:

$$KK_G^{q-1}(\mathbb{R}^{d+m-q,0},\mathbb{R}^{q-p,p-q+m}) = K_G(\mathbb{R}^{9-p-(d+m-q),p-q+m}). \tag{6.4}$$

At first sight the above equation depend on the dimension q of the unstable brane system, and this would rule out our proposal because the Dq-brane is an auxiliary device for the KK-theory formalism, and the result should not depend on it. Then let us argue that this result is indeed independent of q and at the same time we will see that our result is in full agreement with [29]. Remember that from this reference for the Type IIB orbifold with the group $\mathbb{Z}_2$ acting by reflection on n coordinates ($n = 4$ mod 4 in order to preserve some supersymmetry), we say that a Dd-brane is of type (r,s), where $d = r+s$, if it has $r+1$ Neumann directions with $\mathbb{Z}_2$ acting trivially on them and s Neumann directions inverted by $\mathbb{Z}_2$. Then for a given $(r+s)$-brane, the transverse space has dimension $9-(r+s)$; of which $n-s$ directions are inverted under the action of $\mathbb{Z}_2$. Then the K$_G$-theory group classifying Dd-branes in this orbifold is

$$K_{\mathbb{Z}_2}(\mathbb{R}^{n-s,9-n-r}). \tag{6.5}$$

[14] If $d \geq q$ then $m \leq 9-d$.

JHEP12(2008)007

This result is tested by computing these $K_{\mathbb{Z}_2}$-groups and comparing the result with the boundary state formalism, finding full agreement [29].

Though eq. (6.5) is just for $\mathbb{Z}_2$, we will prove that this result is valid for every group acting on the spacetime by means of the spinor representation and then we will argue that the action $\mathbb{Z}_2$ for which (6.5) is valid acts precisely in this way. If we compare our original system with the one described above, we find the following correspondences:

$$n = 9 - p, \qquad d = r + s, \qquad s = d + m - q. \tag{6.6}$$

With these relations one can easily express (6.4) as

$$KK_G^{q-1}(\mathbb{R}^{s,0}, \mathbb{R}^{q-p,p-q+m}) = K_G(\mathbb{R}^{n-s,9-n-r}), \tag{6.7}$$

which is exactly (6.5) with a general group G acting by the spinor representation instead of $\mathbb{Z}_2$. In (6.5) we only assume the existence of the D9-$\overline{\text{D9}}$ unstable system and from (6.7) we see that for each Dq-brane system our proposal is equivalent to (6.5). Then we conclude that the KK_G-formalism is independent of the Dq-system.

Now we argue why the $\mathbb{Z}_2$-action assumed above is spinor. In appendix D we mention that G acts on $\mathbb{R}^n$ through the spinor representation if it acts by a group homomorphism $G \mapsto Spin_n$. By this we mean a homomorphism $\alpha : G \mapsto Spin_n$ such that, when composed with the natural action of $Spin_n$ on $\mathbb{R}^n$ ($x \mapsto \gamma x \gamma^{-1}, \quad x \in \mathbb{R}^n, \quad \gamma \in Spin_n$) we get a representation (which induces an action) of G on $\mathbb{R}^n$.

Consider a D4-brane ($d = 4$) such that $r = 1$ and $s = 3$. Then we can think of the orbifold $\mathbb{Z}_2$-action $x \sim -x$, with $x \in \mathbb{R}^3$ as a π-rotation around some rotation axis in $\mathbb{R}^3$; but we know that each rotation in $\mathbb{R}^3$ can be generated by $\text{SU}(2) \simeq Spin_3$ acting on $\mathbb{R}^3$ through Pauli matrices. In our particular situation, the homomorphism assigning to $-1 \in \mathbb{Z}_2$ the π-rotation U, such that

$$x \mapsto UxU^{-1} = -x, \qquad x \in \mathbb{R}^3$$

does the job. Therefore we can see in this particular example how the $\mathbb{Z}_2$ orbifold action considered in [29] fits in our formalism.

Though it is not easy to find the homomorphism $G \mapsto Spin_n$ for higher values of n, the arguments given above generalize to any n because $Spin_n$ is the double covering of $\text{SO}(n)$, and consequently, to each $\text{SO}(n)$ rotation always correspond at least one element in $Spin_n$.

We have generalized [29] (at least for the case of flat noncompact orbifolds) because our formalism applies to any group action on the spacetime which can be described as a rotation around some axis. Our result also includes some of the examples studied recently in [30], where the orbifold actions are rotations around some axis of the spacetime. In particular, we generalized the flat orbifolds in [30] of the form $\mathbb{Z}_k$ for any $k \in \mathbb{N}$ and $\mathbb{Z}_k \times \ldots \times \mathbb{Z}_k$ (without discrete torsion).

Now we consider an example discussed in [30]. This is the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, with spacetime of the form $\mathbb{R}^4 \times \mathbb{C}^3/\mathbb{Z}_3$, where (x^0, x^1, x^2, x^9) are the coordinates in which $\mathbb{Z}_3$ acts trivially and $z^i = 2^{-\frac{1}{2}}(x^{2i+1} + x^{2i+2})$, $i = 1, 2, 3$ are the coordinates where the generator g of G acts in the form

$$g(z^1, z^2, z^3) \to (\exp(2\pi i v_1) z^1, \exp(2\pi i v_2) z^2, \exp(2\pi i v_3) z^3), \tag{6.8}$$

where $(\upsilon_1, \upsilon_2, \upsilon_3) = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$. The action (6.8) is clearly a rotation in $\mathbb{C}^3$. Therefore the action of $\mathbb{Z}_3$ on any D-brane with s Neumann coordinates in $\mathbb{C}^3/\mathbb{Z}_3$ can be seen as a rotation of the s-coordinates and hence as a spin representation of $\mathbb{Z}_3$ in $Spin_s$ and hence this example is also included in our formalism.

Now we describe the gauge theory living in the unstable Dq-brane system [31]. Following [6], we focus on the stable D-branes which are outside the worldvolume Y of the unstable Dq-brane system and consequently Y can be set to be a point. In our case we need to take into account the images of this point under the action of the group; thus we set Y to be the space of t-points, where t is the cardinality of G. The KK_G-theory group classifying the above D-branes with respect the Dq-unstable system is given by

$$\begin{aligned} KK_G^{q-1}(X, \{t \text{ points}\}) &= KK_G(C_0(X), C(\{t \text{ points}\}) \otimes \mathbb{C}l^{9-q} \\ &= KK_G(C_0(X), (\oplus_{i=1}^t \mathbb{C}) \otimes \mathbb{C}l^{9-q}) = KK_G(C_0(X), \oplus_{i=1}^t (\mathbb{C} \otimes \mathbb{C}l^{9-q})) \\ &= \bigoplus_{i=1}^t KK_G(C_0(X), \mathbb{C} \otimes \mathbb{C}l^{9-q}). \end{aligned} \tag{6.9}$$

Thus we can associate a "gauge group" for each direct summand in (6.9). If we assemble these gauge groups in a block diagonal matrix, we get a matrix M with t blocks and such a matrix belongs to the algebra $\mathbf{B}(C_0(t \quad \text{points})^\infty) \otimes \mathbb{C}l^{9-q}$ of adjointable operators on $C_0(t \quad \text{points})^\infty \otimes \mathbb{C}l^{9-q}$, which is the appropriate Hilbert module for describing $KK_G^{q-1}(X, \{t \quad \text{points}\})$. Moreover, M represents the gauge group of the low energy effective field theory on the Dq-brane worldvolume, which is of the form (for a finite number of branes) $\prod_{i=1}^t \mathrm{U}(N_i)$ or $\prod_{i=1}^t (\mathrm{U}(N_i) \times \mathrm{U}(N_i))$, where $R = \oplus_{i=1}^t N_i r_i$ is the representation of G on the Chan-Paton factors and r_i are the irreps of G. Of course, each block in M is infinite because in order the KK-theory make sense we must assume the presence of an infinite number of Dq-branes [1].

If we take for instance, $q = 7$ then following ref. [6] and the preceding section, the gauge group associated to each of the factors in $\bigoplus_{i=1}^t KK_G(C_0(X), \mathbb{C} \otimes \mathbb{C}l^{9-q}$ is determined by $[\mathbb{C}l^2_{\text{even}}] = \mathbb{C} \oplus \mathbb{C}$ which corresponds to $\mathrm{U}(\infty) \otimes \mathrm{U}(\infty)$. Thus the gauge group of the unstable D7-brane system in the orbifold singularity is (as expected) given by $\prod_{i=1}^t (\mathrm{U}(\infty) \otimes \mathrm{U}(\infty))$.

7. Final remarks

In this paper, we have extended to Type IIB orbifold and Op^--orientifold backgrounds the KK-theory formalism proposed in [1, 6] for Type IIB and Type I string theory respectively.

In particular, for the orientifold case, we considered Op^--planes with $p = 1, 5, 9$, for which the presented formalism naturally incorporates stable D-branes intersecting the orientifold planes, generalizing in this sense the proposal in [13] for the mentioned cases. This is achieved by constructing D-branes from unstable Dq-branes in which the final D-brane has internal and external coordinates with respect to the Dq-brane. In this sense, the internal coordinates are identified with the space Y and the external ones with the space X, where X and Y are the entrances in the KKR-theory bifunctor $KKR(X, Y)$.

Specifically we propose that Dd-branes intersecting Op^--planes are given by the groups in eqs. (3.9) and (3.10)

$$KKR(\mathbb{R}^{d-s-r,0},\mathbb{R}^{(9-q)+(q-p-r),p-s}) = KKR^{1-q}(\mathbb{R}^{d-s-r,0},\mathbb{R}^{q-p-r,p-s}) \qquad \text{for } p<q,$$
$$KKR(\mathbb{R}^{d-s,r},\mathbb{R}^{9-p,p-q+(q+r-s)}) = KKR^{9-2p+q}(\mathbb{R}^{d-s,r},\mathbb{R}^{0,q+r-s}) \qquad \text{for } p>q.$$

In order to show that these groups correctly classify the corresponding configurations of D-branes and orientifolds, we also compute, by extensive use of the Clifford algebras and the structures defined on them, the gauge group and transformation properties of the effective fields living in the worldvolume of the unstable Dq-branes. The transformation properties of the tachyon and scalar fields of this unstable systems are read from the *fixed point Clifford algebra*. This algebra consists of Clifford generators invariant under the involution, determined in turn by the action of the corresponding orientifold plane. The set of algebras related to different configurations is listed in table 5 in text. In all cases we found perfect agreement with Type I T-dual versions, as reported in [6]. This shows that Clifford algebras contain relevant information about stability, RR charge and construction of D-branes in general backgrounds.

However, although this formalism seems powerful enough, the mathematical information in literature concerning other physical relevant cases, as positive RR charged orientifolds, is limited. Working out with expected physical properties instead, we have proposed some KK-theory groups related to the mentioned cases. In particular we have proposed some versions of KK-theory (KKH and KKSp) based on the existence of consistent string theories with D-branes carrying quaternionic and symplectic Chan-Paton bundles. Moreover, we propose, based on their respective K-theories some simple properties of these bifunctors. We also give some clues on how the appropiate structures should be incorporated on the Kasparov modules entering the KKH-theory definition. Similar arguments should apply to KKSp-theory.

As a matter of probe, we have applied this formalism, including the proposal on positive RR charged orientifolds, to elucidate the existence of the so called *exotic* orientifold planes. These planes have been classified by K-theory, but in terms of RR fields. A brane realization of exotic planes reveals a configuration of brane and orientifold planes, for which it is possible to apply the present formalism. For the considered cases, we have found that a brane classification of this planes is possible by means of KK-theory. The results are also in agreement with the RR field classification.

In the orbifold case we reproduce the proposal in [29] in terms of equivariant K_G-theory for a $\mathbb{Z}_2$-orbifold. Moreover, we argued that this prescription is valid for any G-action by means of the spinor representation. In this way our formalism includes every G-action that can be seen as a space-time rotation; including in particular the Type IIB examples considered in [30] of flat orbifolds without discrete torsion (we include an explicit example of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ discussed in this reference and show that $\mathbb{Z}_3$ acts on $\mathbb{C}^3$ by the spinor representation), where the explicit K_G-groups (and hence KK_G-groups) are calculated. We also recover the gauge theory in the unstable Dq-brane systems of Type IIB string theory orbifold spacetimes.

In [25] the K-theory formalism is incorporated to the classification of fluxes in Type IIB string theory which are not sourced by D-branes. It is then natural to incorporate the KK-theory formalism for the classification of these fluxes. Some research in this direction was addressed in [4, 7]

In [32] T-duality is explained in terms of certain isomorphisms of relative K-theory for spacetime compactifications in $\mathbf{T^n}$. So, compactifying the spacetime, amounts to define "relative KK-theory" [33] and the incorporation of T-duality would imply some isomorphisms in the corresponding "relative KK-groups". Some considerations about T-duality and KK-theory has been discussed in another context in [7, 8] (for a recent review see [34]). Finally, in analogy with [35] it should be interesting to incorporate a topologically non-trivial B-field background to the KK-theory classification of D-branes, leading to a twisted KK-theory.

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A. Complex KK-theory

We start by defining a Kasparov module.[15] Let (A, B) be a pair of trivially graded, separable, unital and complex C^*-algebras. An *odd Kasparov A-B module* is a triple $(\mathcal{H}_B, \phi, T)$, where

- $\mathcal{H}_B = B^\infty$ is the Hilbert B-module defined as follows:

$$B^\infty = \left\{ (x_k) \in \prod_{n=1}^{\infty} B \mid \sum_k x_k^* x_k \textit{ converges in } B \right\}.$$

- $\phi : A \to \mathbf{B}(\mathcal{H}_B)$[16] is a unital $*$-homomorphism.

- $T \in \mathbf{B}(\mathcal{H}_B)$ is a self-adjoint operator such that

$$T^2 - 1, \quad [T, \phi(a)] \in \mathcal{K}(\mathcal{H}_B) = B \otimes \mathcal{K} \quad \text{for all } a \in A, \tag{A.1}$$

  where $\mathcal{K}(\mathcal{H}_B)$ is defined such that any pair of elements, $x, y \in \mathcal{H}_B$, gives rise to a map $\Theta_{x,y} : \mathcal{H}_B \to \mathcal{H}_B$ given by $\Theta_{x,y}(z) = x(y, z)$, for all $z \in \mathcal{H}_B$. Then $\mathcal{K}(\mathcal{H}_B)$ is the

[15] Though there are several approaches to Kasparov modules [17, 18, 20], we will use the Fredholm picture which fills out our requirements for physical interpretations.

[16] If E is any Hilbert B-module for a C^*-algebra B, we will denote as $\mathbf{B}(E)$ the set of adjointable operators i.e. the operators $T : E \to E$ such that there exist an operator $T^\dagger : E \to E$ with $(Ta, b) = (a, T^\dagger b)$ for all $a, b \in E$, and (a, b) is the B-valued inner product of $\mathcal{H}_B$ as a Hilbert B-module.

closed linear span of $\{\Theta_{x,y} : x, y \in \mathcal{H}_B\}$ and it is a closed two sided ideal in $\mathbf{B}(\mathcal{H}_B)$. Note that when B is the field of complex numbers, then $\mathcal{K}(\mathcal{H}_B)$ is identified with the space of compact operators on $\mathcal{H}_B$ (denoted $\mathcal{K}$), and $\mathcal{H}_B$ is identified with the space of square summable sequences in the complex numbers.

An *odd Kasparov A-B module* is called *degenerate* if

$$T^2 - 1 = [T, \phi(a)] = 0 \quad \text{for all } a \in A. \tag{A.2}$$

Now, we define some relations on the set of *odd Kasparov A-B modules*:

- Two triples $(\mathcal{H}_B, \phi_0, T_0)$ and $(\mathcal{H}_B, \phi_1, T_1)$ are called *unitarily equivalent* if there exists an unitary operator $U \in \mathbf{B}(\mathcal{H}_B)$ with $T_0 = U^* T_1 U$ and $\phi_0(a) = U^*\phi_1(a)U$ for all $a \in A$.

- Let $(\mathcal{H}_B, \phi_i, T_i)$ be *odd Kasparov A-B modules* for $i = 0, 1$; let (E, ϕ, T) be an *odd Kasparov A-B $\otimes$ C[0, 1] module* and let $f_t : B \otimes C[0, 1] \to B$ denote the evaluation map $f_t(g) = g(t)$. Then $(\mathcal{H}_B, \phi_0, T_0)$ and $(\mathcal{H}_B, \phi_1, T_1)$ are called *homotopic* and (E, ϕ, T) is called a *homotopy* if $(E \otimes_{f_i} B, f_i \circ \phi, f_{i_*}(T))$ is unitarily equivalent to $(\mathcal{H}_B, \phi_i, T_i)$, $i = 0, 1$, where $f_{i_*}(T)(a) := f_i(T(a))$.

- If $E = C([0, 1], \mathcal{H}_B)$ and for all $a \in A$ the induced maps $t \to T_t, t \to \phi_t(a)$ are strongly $*$-continuous, then (E, ϕ, T) is called a *standard homotopy.* When in addition ϕ_t is constant and T_t is norm continuous then we say that (E, ϕ, T) is an *operator homotopy.*

- In the definition of the group $KK^1(A, B)$, two *odd Kasparov modules* $(\mathcal{H}, \phi_i, T_i)$, $i = 0, 1$ are defined to be equivalent (and denoted $\sim_{oh}$) if there are degenerate Kasparov modules $(\mathcal{H}_B, \phi'_i, T'_i)$, $\quad i = 0, 1$ such that $(\mathcal{H}_B \oplus \mathcal{H}_B, \phi_i \oplus \phi'_i, T_i \oplus T'_i)$, $\quad i = 0, 1$ are operator homotopic up to unitary equivalence. Then $KK^1(A, B)$ is the set of equivalence classes of odd Kasparov modules under $\sim_{oh}$.

The other KK-group $KK(A, B) \equiv KK^0(A, B)$ is the set of equivalence classes of $\mathbb{Z}_2$-graded triples $(\widehat{\mathcal{H}}, \widehat{\phi}, F)$, called *even Kasparov A-B modules*, with

$$\widehat{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_1, \qquad \widehat{\phi} = diag(\phi_0, \phi_1), \qquad F = \begin{pmatrix} 0 & T^\dagger \\ T & 0 \end{pmatrix}, \tag{A.3}$$

where $\mathcal{H}_i$ $(i = 0, 1)$ are Hilbert B-modules, $\phi_i : A \to B(\mathcal{H}_i)$ is a unital $*$-homomorphism for $i = 0, 1$ and $T \in \mathbf{B}(\mathcal{H}_0, \mathcal{H}_1)$ is an adjointable operator such that

$$T^\dagger T - 1, \qquad TT^\dagger - 1, \quad T\phi_0(a) - \phi_1(a)T \in B \otimes \mathcal{K} \quad \text{for all } a \in A. \tag{A.4}$$

The grading is induced by the standard even grading operator $diag(1, -1)$ where we identify $\mathbf{B}(\hat{\mathcal{H}}_B) = M_2(M(B \otimes \mathcal{K})) = M(B \otimes \mathcal{K})$, where $M(B \otimes \mathcal{K})$ is the multiplier algebra of $B \otimes \mathcal{K}$ and $M_2(A)$ is the C^*-algebra of 2×2 matrices with entries in the C^*-algebra A.

The group $KK^0(A, B)$ is defined as the set of equivalence classes of even Kasparov modules with the equivalence relation $\sim_{oh}$ defined above.

It can be proved that $KK^1(A,B) = KK^0(A, B \otimes \mathbb{C}l^1)$ with $\mathbb{C}l^1$ being the complex Clifford algebra generated by $\{1, e_1\}$, where $e_1^2 = 1$. This is the approach we will take for the definitions of the real, Real and equivariant KK-groups for real, Real C^*-algebras and G-algebras. In general, one can define higher KK-groups as $KK^n(A,B) = KK(A, B \otimes \mathbb{C}l^n)$, but periodicity mod 2 tells us that the only KK-theory groups are KK^0 and KK^1.

B. Orthogonal (real) KK-theory

The real KKO-group is defined similarly as above, but substituting complex objects by real ones (real C^*-algebras, real Hilbert B-modules, etc.). In other words, the structures are defined over the field of the real numbers instead of the complex numbers field.

To define the higher real KKO-groups we need to define a real C^*-algebra structure in the real Clifford algebra $Cl^{n,m}$, where $Cl^{n,m}$ is generated (as an algebra over $\mathbb{R}$) by $\{e_i \in \mathbb{R}^{n+m}\}_{i=1,\ldots,n+m}$ with the relations

$$\begin{aligned} e_ie_j + e_je_i &= 0 && (i \neq j), \\ e_i^2 &= -1 && (i = 1, \ldots, n), \\ e_i^2 &= 1 && (i = n+1, \ldots, n+m). \end{aligned} \tag{B.1}$$

The C^*-algebra involution is defined on the generators $\{e_i \in \mathbb{R}^{n+m}\}$ as follows:

$$\begin{aligned} e_i^* &= -e_i && (i = 1, \ldots n), \\ e_i^* &= e_i && (i = n+1, \ldots, n+m), \\ (e_1 \cdots e_l)^* &= e_l^* \cdots e_1^*, \end{aligned} \tag{B.2}$$

and extending it by linearity.

It can be shown [17, 20] that $KKO(A \otimes Cl^{n,m}, B \otimes Cl^{r,s})$ depends only on $(m-n) - (s-r)$, so we can define with no ambiguity:

$$KKO_{m-n+r-s}(A,B) = KKO(A \otimes Cl^{n,m}, B \otimes Cl^{r,s}). \tag{B.3}$$

Thus, for $n \in \mathbb{Z}$ we define

$$KKO^{-n}(A,B) = KKO_n(A,B) = \left\{ \begin{array}{ll} KKO(A, B \otimes Cl^{n,0}) & n > 0 \\ KKO(A, B \otimes Cl^{0,-n}) & n < 0 \end{array} \right\}. \tag{B.4}$$

The KKO_n-groups are periodic mod 8.

Both, complex and real KK-theories share the following crucial property called Bott Periodicity:

$$KKO^k(X,Y) = KKO^{k-n}(X \times \mathbb{R}^n, Y) = KKO^{k+m}(X, Y \times \mathbb{R}^m), \tag{B.5}$$

where $\mathbb{R}^n$ stands for $C_0(\mathbb{R}^n)$. In general, for any locally compact topological spaces X and Y, we denote $KKO^n(X,Y) \equiv KKO^n(C_0(X), C_0(Y))$, where $C_0(X)$ $(C_0(Y))$ is the C^*-algebra of continuous real (or complex when we are dealing with KK-groups) valued functions on X (Y), vanishing at infinity.

JHEP12(2008)007

B.1 KK-theory applied to D-branes

Suppose we have a spacetime of the form $X \times Y$ (with $dimY = q+1$) and an unstable system of an infinite number of Dq-branes extended on Y, in Type II string theory. It was proposed in [1] that the solitonic configurations (which turn out to be D-branes) of this system are classified by the complex KK-groups: The $KK^1(X,Y)$-group for non-BPS Dq-branes and $KK^0(X,Y)$-group for a Dq- $\overline{\text{D}q}$ system ($\overline{\text{D}q}$ denotes an anti Dq-brane) of stable Dq branes. The grading for the even Kasparov modules described in the definition of the KK^0-group is associated with the Dq- $\overline{\text{D}q}$-branes.

Now, we will review the way KKO-groups are applied to classify D-branes in Type I string theory.

In [12] it is argued that the K-theory group classifying Dd-brane charges inside the worldvolume of an unstable Dq-brane system in Type I string theory is the real K-theory group $KO^{q-1}(Y)$, where Y is the worldvolume manifold of the unstable system. The proposal in [6] is that the group that correctly classifies D-branes stretched along both longitudinal and transverse directions to the unstable Dq-brane system is $KKO^{q-1}(X,Y)$ where X and Y have the same meaning that in the complex case.

The elements of $KKO^0(X,Y)$ can be interpreted in the same way as for Type II string theories in terms of Dq- $\overline{\text{D}q}$-brane system (for $q = 0, 9$) wrapped in Y.

In general $KKO(C_0(X), C_0(Y) \otimes Cl^{n,m})$ consists of equivalence classes of triples $(\widehat{\mathcal{H}}, \widehat{\phi}, F)$ where $\widehat{\mathcal{H}} = C_0(Y)^{\infty} \otimes Cl^{n,m}$, $\widehat{\phi} : C_0(X) \to \mathbf{B}(\widehat{\mathcal{H}})$ is a $*$-homomorphism, and F is a self-adjoint operator in $\mathbf{B}(\widehat{\mathcal{H}}) = \mathbf{B}(C_0(Y)^{\infty} \otimes Cl^{n,m})$. We have the additional requirement that $\widehat{\phi}(a)(a \in C_0(X))$ be even and F odd with respect to the $\mathbb{Z}_2$-grading. In this way, we can write:

$$F = \sum_{v_r \in Cl^{n,m}_{\text{odd}}} T_r \otimes v_r, \qquad \widehat{\phi}(a) = \sum_{w_r \in Cl^{n,m}_{\text{even}}} \Phi_r \otimes w_r, \tag{B.6}$$

where $T_r, \Phi_r \in \mathbf{B}(C_0(Y))$, w_r and v_r span the sets of even an odd elements in $Cl^{n,m}$ denoted as $Cl^{n,m}_{\text{even}}$ and $Cl^{n,m}_{\text{odd}}$.

B.2 Field content in unstable type I non-BPS branes

As it was studied in [6], the field content representation of an unstable non-BPS Dq-brane in Type I theory can be elucidated from the real Clifford algebra since in this case, as can be seen from B.1, the tachyon and scalar fields satisfy some requirements. The tachyon field is an odd self-adjoint operator, while ϕ is an even self-adjoint map. This fixes their representations under the gauge transformation, which is an even unitary transformation on the Hilbert space $\mathcal{H}$. For completness we reproduce the results obtained in [6] in table 7.

For Dq-branes in orientifold backgrounds like those studied in this paper ($O1^-$ and $O5^-$) it suffices to compute the fixed point algebra $({}^{p}\mathbb{C}l^{n,0})_{\text{fix}}$ or $({}^{p}\mathbb{C}l^{0,n})_{\text{fix}}$. These fix point algebras are in general of the form $Cl^{n,0}$ or $Cl^{0,n}$ (with the only exception of algebras related to non-BPS D1 and D9 branes). This algebra fixes in turn the field content and gauge group for each case. This was explicity done in [6] which results are summarized in table 7.

Dq	$Cl^{n,m}$	ϕ	T	Gauge group
D(-1) D7	$Cl^{2,0}$	*adj.*	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	U(∞)
D0 D8	$Cl^{1,0}$	$\square\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$O(\infty)$
D1 D9	$Cl^{1,1}$	$(1,\square\square),(\square\square,1)$	$(\square,\bar{\square})$	$O(\infty)\times O(\infty)$
D2	$Cl^{0,1}$	$\square\square$	$\square\square$	$O(\infty)$
D3	$Cl^{0,2}$	*adj.*	$\square\square$	U(∞)
D4	$Cl^{0,3}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square\square$	Sp(∞)
D5	$Cl^{4,0}$	$(1,\begin{smallmatrix}\square\\\square\end{smallmatrix}),(\begin{smallmatrix}\square\\\square\end{smallmatrix},1)$	$(\square,\bar{\square})$	Sp(∞) $\times$ Sp(∞)
D6	$Cl^{3,0}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	Sp(∞)

Table 7: Field content of unstable non-BPS Dq-branes in Type I theory and the relevant real Clifford algebra, as obtained in Reference [6].

C. Clifford algebras and real KKR-theory

In this appendix we will describe the additional structure in the Clifford algebras which is necessary for the definition of the KKR bifunctor.

C.1 Even and odd parts of the real Clifford algebra

In the Clifford algebra $Cl^{n,m}$ there is a natural grading induced by the map $\alpha : Cl^{n,m} \mapsto Cl^{n,m}$ acting on the generators like $\alpha(e_i) = -e_i$ for all $i = 1, \ldots, n+m$. Then, α is extended to the whole Clifford algebra by linearity. In this way, the real Clifford algebra splits in even and odd parts, defined as the eigenspaces with eigenvalues 1 and -1 respectively, i.e.

$$Cl^{n,m} = (Cl^{n,m})_{\text{even}} \oplus (Cl^{n,m})_{\text{odd}}, \tag{C.1}$$

where $a \in (Cl^{n,m})_{\text{even}}$ if $\alpha(a) = a$ and $a \in (Cl^{n,m})_{\text{odd}}$ if $\alpha(a) = -a$.

A general element of $Cl^{n,m}$ can be written as

$$\alpha = a^{i_1} e_{i_1} + a^{i_1 i_2} e_{i_1 i_2} + \cdots + a^{1\cdots(n+m)} e_{1\cdots(n+m)} = \sum_{k=1}^{n+m} \sum_{i_1<i_2<\cdots<i_k} a^{i_1\cdots i_k} e_{i_1\cdots i_k}, \tag{C.2}$$

where $e_{i_1\ldots i_k} \equiv e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k}$.

It is easy to show that the even and odd parts can be expressed in the following way:

$$\begin{aligned} (Cl^{n,m})_{\text{even}} &= \left\{\alpha \in Cl^{n,0} | \alpha = a^0 + a^{i_1 i_2} e_{i_1 i_2} + a^{i_1 i_2 i_3 i_4} e_{i_1 i_2 i_3 i_4} + \ldots \right\}, \\ (Cl^{n,m})_{\text{odd}} &= \left\{\alpha \in Cl^{n,0} | \alpha = a^{i_1} e_{i_1} + a^{i_1 i_2 i_3} e_{i_1 i_2 i_3} + \ldots \right\}. \end{aligned} \tag{C.3}$$

From now, we will concentrate on the algebras $Cl^{n,0}$.[17]

[17] Notice that for the complex Clifford algebra $\mathbb{C}l^n$, we have $\mathbb{C}l^n = Cl^{n,0} \otimes \mathbb{C} \cong Cl^{0,n} \otimes \mathbb{C}$. Then all expressions and facts in this appendix are valid for the analog ones for $Cl^{0,n}$.

The grading in the Clifford algebra $Cl^{n,0}$ induce a grading in the complex Clifford algebra $\mathbb{C}l^n$ as follows:

$$\begin{aligned}\mathbb{C}l^n &= Cl^{n,0} \otimes \mathbb{C} \\ &= \big((Cl^{n,0})_{\text{even}} \oplus (Cl^{n,0})_{\text{odd}}\big) \otimes \mathbb{C} \\ &= ((Cl^{n,0})_{\text{even}} \otimes \mathbb{C}) \oplus ((Cl^{n,0})_{\text{odd}} \otimes \mathbb{C}) \\ &\equiv \mathbb{C}l^{n,0}_{\text{even}} \oplus \mathbb{C}l^{n,0}_{\text{odd}}. \end{aligned} \tag{C.4}$$

Hence, an element a in the even part of $\mathbb{C}l^n \equiv \mathbb{C}l^{n,0}$ is just written as $\alpha \oplus (\beta \otimes i)$ with $\alpha, \beta \in Cl^{n,0}_{\text{even}}$, and a similar expression for the odd part.

C.2 Fixed point algebra

An element a in the complexified Clifford algebra $\mathbb{C}l^n = Cl^{n,0} \otimes \mathbb{C}$ is given by

$$a = \alpha \oplus (\beta \otimes i), \tag{C.5}$$

where α and β are elements of the real Clifford algebra $Cl^{n,0}$.

Suppose that there is a Real involution defined on $\mathbb{C}l^n$, such that it is now a Real algebra denoted ${}^p\mathbb{C}l^{n,0}$ (as explained in section 3.2, the involution is given by the action of the orientifold plane on the generators of the algebra and extended by linearity).

On the other hand we have

$${}^p\mathbb{C}l^{n,0} = {}^p\left(Cl^{n,0} \otimes \mathbb{C}\right) = {}^pCl^{n,0} \otimes \overline{\mathbb{C}}, \tag{C.6}$$

where $\overline{\mathbb{C}}$ denotes the field of complex numbers with Real involution defined by usual complex conjugation and ${}^pCl^{n,0}$ denotes the Clifford algebra $Cl^{n,0}$ with some Real involution (again, this involution is determined by the orientifold plane on the generators of the algebra and extended by linearity). Then it is enough to select the proper involution on the generators of $Cl^{n,0}$ to know the involution on ${}^p\mathbb{C}l^{n,0}$.

The fixed point algebra of ${}^p\mathbb{C}l^{n,0}$ is defined as the set of elements in ${}^p\mathbb{C}l^{n,0}$ which are invariant under the involution,[18] i.e.

$$\left({}^p\mathbb{C}l^{n,0}\right)_{\text{fix}} = \left\{a \in {}^p\mathbb{C}l^{n,0} |\ a = \overline{a} \equiv \mathcal{I}_{9-p}(a)\right\}, \tag{C.7}$$

where $\mathcal{I}_{9-p}(i) = -i$. Hence an element of the fixed point algebra must satisfy the following constraint

$$\overline{\alpha} \oplus (\overline{\beta} \otimes \overline{i}) = \overline{\alpha} \oplus (\overline{\beta} \otimes (-i)) = \overline{\alpha} \oplus ((-\overline{\beta}) \otimes i) = \alpha \oplus (\beta \otimes i). \tag{C.8}$$

As a trivial example, consider $p = 9$. Then for any unstable Dq-brane system,[19] we have $q \leq p$. Following the criteria explained in section 4.1, we define the involution to be the trivial one in each generator of the relevant Clifford algebra $Cl^{0,q-1}$. Then, by (C.8) we identify

$$\left({}^9\mathbb{C}l^{0,q-1}\right)_{\text{fix}} = Cl^{0,q-1}. \tag{C.9}$$

[18] This definition applies to any algebra with some Real involution.

[19] For simplicity, suppose $q > 2$.

This is expected since in an $O9^-$-plane background, i.e. in Type I theory, the whole nine-dimensional space is a fixed point under the orientifold involution and hence, D-branes are characterized by orthogonal Clifford algebra, as shown in [6].

As explained in section 4, the tachyon, the scalar fields and the gauge transformation on the unstable Dq-brane system all belong to the fixed point algebra and to some of the even or odd parts of ${}^9\mathbb{C}l^{n,0}$ for some n. It turns out that for selecting an element with some of these properties, it is enough to compute the fixed point algebra (as explained above), which will be isomorphic to a real Clifford algebra $Cl^{r,s}$ for some r and s.[20] Then we just need to compute the natural even and odd part of $Cl^{r,s}$ as a Clifford algebra as explained in C.1.

D. Equivariant KK-theory

In this appendix we will describe the pertinent modifications to the KK-theory bifunctor described earlier to define the equivariant KK_G-theory, which turns out to be the appropiate tool for the classification of D-branes in orbifold singularities. A C^*-algebra A is called a G-algebra if there is a compact group G acting on it by the automorphism group, i.e., by a map $\alpha : G \mapsto Aut(A)$. In this appendix all C^*-algebras are required to be G-algebras.

The G-action is said to be continuous if $\alpha : G \mapsto Aut(A)$ is continuous. This definition is rephrased by requiring that the induced map $G \times A \mapsto A : (g, a) \mapsto g(a)$ is norm continuous, where A is realized as an operator algebra with the strong operator topology.

By the Hilbert G-module $\mathcal{H}_B$ we mean the Hilbert B-module $\mathcal{H}_B$ together with a linear action of G, such that:

- $g(xb) = g(x)g(b) \qquad \text{for all} \quad g \in G, \quad x \in \mathcal{H}_B, \quad b \in B,$
- $(g(x), g(y)) = g((x, y)) \qquad \text{for all} \quad g \in G, \quad x, y \in \mathcal{H}_B,$

where (x, y) is the B-valued inner product of $\mathcal{H}_B$ as a Hilbert B-module. We have the additional condition that this action be norm-continuous i.e the map $g \mapsto \|(gx, gx)\|$, $x \in \mathcal{H}_B$ is norm continuous in the strong operator topology. An element $x \in \mathcal{H}_B$ is said to be invariant if $g(x) = x$ for all $g \in G$.

In $\mathbf{B}(\mathcal{H}_B)$ there is an induced natural action as follows: If $F \in \mathbf{B}(\mathcal{H}_B)$, then $g(F)$ is defined as $(g(F))(x) = g(F(g^{-1}(x))), \quad g \in G, \quad x \in \mathcal{H}_B$. This induced G-action is not norm continuous in general and those F for which this holds are called G-continuous. They make up a C^*-subalgebra of $\mathbf{B}(\mathcal{H}_B)$ which contains $\mathcal{K}(\mathcal{H}_B)$.

Now, consider all even Kasparov G-modules $(\mathcal{H}_B, \phi, F)$, i.e. the set of even Kasparov modules such that:

- $\mathcal{H}_B$ is a G-Hilbert B-module.
- $\phi : A \mapsto \mathbf{B}(\mathcal{H}_B)$ is an equivariant $*$-homomorphism.
- $F \in \mathbf{B}(\mathcal{H}_B)$ is an invariant (in particular, G-continuous) element.

[20] Notice that on $Cl^{r,s}$ there is not a Real involution anymore and both r and s depend on the Real involution defined in ${}^p\mathbb{C}l^{n,0}$ or equivalently in $Cl^{n,0}$.

Then the equivariant KK_G-group, denoted $KK_G(A,B)$ is defined as the set of equivalence classes of even Kasparov G-modules under the equivalence relation $\sim_{oh}$ defined exactly as in appendix A, but with the additional condition that the operator $U \in \mathbf{B}(\mathcal{H}_B)$ generating the *unitarily equivalence* relation be invariant under the G-action.

The higher KK_G-groups $KK^n_G(A,B)$ are defined as before, i.e $KK^n_G(A,B) = KK^n_G(A, B \otimes \mathbb{C}l^n)$ where G acts trivially on $\mathbb{C}l^n$ and again they have periodicity mod 2.

The KK_G-groups share the analog of the property (B.5) of KK-groups:

$$KK^k_G(X,Y) = KK^{k-n}_G(X \times \mathbb{R}^n, Y) = KK^{k+n}_G(X, Y \times \mathbb{R}^n), \tag{D.1}$$

with the additional requirement that G acts on $\mathbb{R}^n$ by means of the spinor representation, i.e. by a group homomorphism $G \mapsto Spin_n$.

One of the most important properties shared by all versions of Kasparov K-theory is additivity:

$$KK_G(A, B_1 \oplus B_2 \oplus \cdots \oplus B_n) = KK_G(A, B_1) \oplus KK_G(A, B_2) \oplus \cdots \oplus KK_G(A, B_n). \tag{D.2}$$

A similar expression also holds for the first of the entries in the KK_G-functor.

We mention it here because it is particularly important for calculating the low energy effective gauge theory living in the worldvolume of an unstable D-brane system placed at an orbifold singularity.

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JHEP12(2008)007

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